AD

USAAVLABS TECHNICAL REPORT 66-44

ANALYSIS OF THE DYNAMICS OF SUPERCRITICAL SHAFTS ON MANY FLEXIBLE SUPPORTS BY USING TRANSMISSION LINE ANALOGY

By

Y. N. Liu

J. A. Friedericy

September 1966

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

Purchase Order AMC(T)-01-66-16

UNIVERSITY OF VIRGINIA

CHARLOTTESVILLE, VIRGINIA

U S ARMY MATERIEL COMMAND

Disclaimers

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission, to manufacture, use, or sell any patented invention that may in any way be related thereto.

Disposition Instructions

Destroy this report when no longer needed. Do not return it to originator.

DEPARTMENT OF THE ARMY
U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA 23604

This report is published in the interest of providing additional information on the behavior of shafts operating at supercritical speeds.

The rigorous analysis presented herein, which accounts for the effects of rotating inertia, gyroscopic motion, and shear deformation, may explain some apparent discrepancies that have resulted from the use of the simplifying assumptions in the work done by the Battelle Memorial Institute. The latter work was reported in ASD-TDR 62-728, "Design Criteria for High-Speed Power-Transmission Shafts", under Air Force and USAAVLABS sponsorship.

Task 1M121401D14414
Purchase Order AMC(T)-0I-66-16
USAAVLABS Technical Report 66-44
September 1966

ANALYSIS OF THE DYNAMICS OF SUPERCRITICAL SHAFTS ON MANY FLEXIBLE SUPPORTS BY USING TRANSMISSION LINE ANALOGY

by

Y. N. Liu
J. A. Friedericy

Prepared by

University of Virginia
School of Engineering and Applied Science
Charlottesville, Virginia

for

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

FOREWORD

The following constitutes a technical report concerned with research conducted by the School of Engineering and Applied Science, University of Virginia, Charlottesville, Virginia, as an extension of work initiated on a grant awarded to the University of Virginia by the U. S. Army Aviation Materiel Laboratories, Fort Eustis, Virginia, under Grant No. DA-AMC-44-177-63-G10. The authors express their appreciation and thanks to the U. S. Army Aviation Materiel Laboratories for providing the funds for typing the manuscript and publishing the report under Task 1M121401D14414, Purchase Order AMC(T)-01-66-16. They would also like to express their gratitude to Dr. R. T. Eppink for his invaluable suggestions and Mrs. Ann Symmers and Mrs. Barbara Little for typing the final manuscript.

ABSTRACT

This report presents an analysis of the dynamics of supercritical shafts on many flexible supports in terms of a so-called "transmission line analogy". The primary objective of the study is to develop a direct analytical approach for optimizing the support conditions, in terms of minimum flexural vibration behavior, for supercritical shafts flexibly supported on mass-spring-damper units at different locations along the shaft. The approach is based on the traveling wave concept as used in electrical transmission line theory. The governing differential equation used in this analogy includes terms which account for the effects of rotating inertia, gyroscopic motion, and shear deformations.

If the solution of the governing differential equation is manipulated by means of rather complicated matrix algebra, the dynamic response of the rotating shaft can be expressed in traveling wave form, which leads to the criterion for impedance matching and optimized support conditions. The impedance at each supporting location corresponding to minimum vibration response must equal the characteristic impedance of the shaft. This condition is termed a "matched" condition.

A weaker than optimum form of impedance matching is the "quasi-matched" condition, in which only the predominant term of the reflection matrix for a support is made to vanish.

A rotating shaft with three supports is used to illustrate the matched impedance concept for determining optimum support conditions for multisupported hypercritical shafts.

The study has led to the following specific conclusions:

1. The transmission line analogy can be extended to shafts having any number of interior supports. However, the solution is considerably more complicated for shafts having more than one interior support.

2. For the shaft with both end impedances matched with the characteristic impedance of the shaft, no intermediate support is needed to assist in the minimization of vibration response.

3. If it is physically impossible or impractical to terminate a shaft in its characteristic impedance, quasi-matched end impedances, or quasi-matched interior supports when end conditions are not available for optimization, should provide good performance.

4. When the shaft and end support impedances are not matched and only one interior support is used, two approaches may be employed to assist in the minimization of vibration response:

a. If both ends have the same configuration, the matched interior support may be placed at the mid-span.

b. If one end support is different from the other, the closer the matched intermediate support is placed to one of the ends, the more effectively it will minimize the vibration response of the shaft.

5. When the shaft and end support impedances are not matched, the use of two matched intermediate supports placed closely to the ends of the shaft is recommended.

CONTENTS

ILLUSTRATIONS

Figure		Page

SYMBOLS

a	=	Minimum X_1 coordinate of A
a_{jk}	=	Distance from left end to the k^{th} driving force in the j^{th} span
A	=	Elastic curve of A_0, or area of the shaft cross section
A_0	=	Left end body
ASB	=	A_0, S_0, B_0, collectively
b	=	Maximum X_1 coordinate of B
B	=	Elastic curve of B_0
B_0	=	Right end body
c_s	=	Velocity of sound for the shaft material
C_n	=	n-dimensional vector space over a field of complex numbers, usually n = 2
$\overset{\#}{C}_{11}$, $\overset{\#}{C}_{12}$, $\overset{\#}{C}_{21}$, $\overset{\#}{C}_{22}$, $\overset{\#}{C}_{+}$, $\overset{\#}{C}_{-}$	=	2×2 matrices with complex entries, functions of ω; defined in Appendix B
$\overset{\#}{C}_y$, $\overset{\#}{C}_f$	=	2×2 diagonal matrices with complex entries, functions of e_1, e_2, e_3; defined in Appendix B
C_T	=	Translational damping coefficients
C_R	=	Rotational damping coefficient
$d(x)$	=	ASB whose X_1 coordinate at rest is x
e_1, e_2, e_3	=	Functions of ω, ω_0, e'; defined in Appendix B
e'	=	$(1/K')(E_y/E_s)$
E_s	=	Shear modulus for shaft material
E_y	=	Young's modulus for shaft material
F	=	Normalized internal force vector (column matrix with complex entries)

F_{11}, F_{12}	=	Normalized components of internal force along the X_2, X_3 axes, respectively
F_{21}, F_{22}	=	Normalized components of internal moment along $-X_3$, X_2 axes, respectively; a right-hand screw sign convention is used for moments
$H(p)$	=	$\overset{\#}{0}$ when $p < 0$, = $\overset{\#}{I}$ when $p > 0$
i	=	$\sqrt{-1}$, or subscript indicating the i^{th} span and i^{th} support
I	=	Moment of inertia of the shaft cross-sectional area about a diameter
$\overset{\#}{I}$	=	2×2 identity matrix
I_m	=	Mass moment of inertia of the mass element about a diameter
I'_m	=	Mass moment of inertia of the mass element about its center of gravity
j	=	Subscript indicating the j^{th} support or the j^{th} span
k	=	Subscript indicating the k^{th} driving force in a certain span
K'	=	Shape factor for shaft cross section
K_T	=	Translational spring constant
K_R	=	Rotational spring constant
ℓ_n	=	Shaft length
ℓ_j	=	Coordinate on X_1-axis of the j^{th} support; $j = 1, 2, \ldots, n$
M	=	Mass in the support configuration
$\overset{\#}{M}$	=	Mobility
p	=	Arbitrary real number
$\bar{P}(a_{jk})$	=	Normalized single driving force vector at location $x = a_{jk}$ (column matrix with complex entries)

$\bar{Q}$	=	Normalized force vector (column matrix with complex entries)
$\bar{q}_j$, $\bar{r}_j$	=	Arbitrary constant vectors
$\overset{\#}{R}(x)$	=	Propagation matrix of a traveling wave evaluated at x
R_b	=	Radius of gyration in bending
s	=	Laplace transform variable, $x = i\omega$ for steady-state solution
S	=	Elastic curve of shaft, S_0
S_0	=	Shaft
$\overset{\#}{S}(\gamma)$	=	$\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$, γ is a real number
t	=	Time coordinate, or subscript indicating partial differentiation with respect to variable t
$\overset{>}{\bar{U}}(x)$, $\overset{<}{\bar{U}}(x)$	=	Incident wave functions, respectively, traveling to the right and left
$\overset{\gg}{\bar{U}}(x)$, $\overset{\ll}{\bar{U}}(x)$	=	Reflection wave functions, respectively, traveling to the right and left
$\overset{>>>}{\bar{U}}(x)$, $\overset{<<<}{\bar{U}}(x)$	=	Total wave functions, respectively, traveling to the right and left
W	=	Weight of a differential mass element of the shaft
x	=	X_1-coordinate, or subscript indicating partial differentiation with respect to variable x
X_1, X_2, X_3	=	Normalized right-hand orthogonal coordinate axes
X_1X_2-plane	=	Real plane
X_1X_3-plane	=	Imaginary plane
$\bar{Y}$	=	Normalized position vector (column matrix with complex entries)

Y_{11}, Y_{12} = Normalized projections of the deflections of elastic curve of shaft on the X_1X_2– and X_1X_3-planes, respectively

Y_{21}, Y_{22} = Normalized projections of the angles of inclination of elastic curve of shaft on the X_1X_2– and X_1X_3-plane, respectively

$\overset{\#}{z}$ = $\frac{1}{\sqrt{\omega}} F\overset{\#}{Z}F$, $F = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\omega} \end{bmatrix}$

$\overset{\#}{Z}$ = $E\overset{\#}{Z}E$, $E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\overset{\#}{Z}$ = Impedance

$\overset{\#}{Z}_j(\ell_j)$ = The j^{th} support impedance

$\overset{\#}{Z}_{j\ell}(x)$ = Impedance looking to the left in the j^{th} span at the location x

$\overset{\#}{Z}_{jr}(x)$ = Impedance looking to the right in the j^{th} span at the location x

$\overset{\#}{Z}_s$ = Characteristic impedance of the rotating shaft

$\bar{0}$ = Null vector

$\overset{\#}{0}$ = Null matrix

-1 = Superscript indicating the inverse of a matrix

$\overset{\#}{\Gamma}_{j\ell}(x)$ = Reflection matrix looking to the left in the j^{th} span at the location x

$\overset{\#}{\Gamma}_{jr}(x)$ = Reflection matrix looking to the right in the j^{th} span at the location x

ω	=	Angular velocity of rotating driving force or the frequency of the flexural vibration of the shaft
ω_0	=	Angular velocity of the rotating shaft about X_1-axis
ρ	=	Mass density of the shaft material
$[\alpha, \beta]$	=	Closed interval with end points, α, β on the X_1-axis
(α, β)	=	Open interval with end points, α, β on the X_1-axis
$\sim$	=	Indicating a Laplace transform quantity
#	=	Indicating a 2×2 matrix with complex entries
'	=	Superscript indicating the transpose of a matrix
*	=	Superscript indicating an unnormalized variable

CHAPTER 1

INTRODUCTION

OBJECTIVES

For any rotating shaft, there exists a series of discrete speeds at which centrifugal forces resulting from mass unbalances cause progressively greater shaft deflections. The elastic restoring forces developed as the shaft deflects are overcome by these ever-increasing centrifugal forces. Extremely large deflections and even destruction of the shaft and its bearings can result from operation at these speeds, called critical speeds. For this reason, designers of power-transmission equipment normally avoid the problem by operating shafts below their first critical speed.

There are, of course, disadvantages to restricting operation to below the first critical speed. For transmitting a given horsepower, torque and, consequently, shaft size must be increased as operating speed is reduced. In the case of long shafts, the shaft size must be increased above the size required to transmit the torque simply to raise the first critical speed above the operating speed range; alternatively, the shaft size may be determined by the torque loading, but additional bearings must be installed to support the shaft and thereby to raise its first critical speed. The major disadvantage of these conventional practices, especially as applied to aircraft, is the weight penalty.

It has been shown that shafting can be operated consistently far above its first critical speed, with consequent savings in shaft and support weight (references 5, 11, and 21). In short, supercritical-speed shafting, with its associated advantage in weight, is a very practical and feasible means of transmitting power. With today's high-speed power sources, it is especially attractive, since considerable weight could be pared from engines and bearings by transmitting power at the same speed as it is produced.

Transmitting large horsepowers with small-diameter shafts presents the problem of controlling shaft vibration at the critical speeds. Successful operation of flexible shafts is usually achieved by balancing to reduce dynamic forces and also by introducing support conditions (spring, damping, mass) which tend to minimize runout amplitudes and/or bearing loads at the important critical frequencies (references 11 and 21). Both of these techniques should be employed simultaneously to bring about smooth shaft operation through the critical speeds. Although the provision of appropriate spring, damping, and mass coefficients (impedance) at the supports, alone, may permit the rotor to negotiate the critical speeds in an acceptable manner, the ease with which this may be accomplished will be greater for the better balanced (less crooked) shaft. Balancing of a supercritical shaft may be achieved by either the proper attachment of counterweights or the placement of greater restrictions on the fabrication tolerances of the shaft.

From a theoretical point of view, different mathematical formulations (reference 8) have to be developed to implement the two techniques. The analysis of a supercritical shaft supported at various points along the length of the shaft by flexible damping bearings is represented with good accuracy by the steady-state solution of the equation of motion for the beam vibrating in two mutually perpendicular planes. This solution has been used to simulate the performance of the rotating shaft under actual running conditions in which the shaft defects (unbalance, and initial crookedness) provide distributed forcing functions. The approach provides a means to study the effects of imperfections and balancing on shaft performance.

On the other hand, the solution can be interpreted also in terms of "a transmission line analog". This approach is based on the recognition of the existence of an analog between the amplitude response of the vibrating beam and the voltage amplitudes in electrical transmission lines. The shaft runout is treated as a series of deflection waves (voltage waves) traveling along the shaft (transmission line). These waves are in part absorbed and in part reflected at the supports (loads). In other words, the dynamic responses are expressed in traveling wave form along the shaft in a manner analogous to the treatment of electrical response waves in transmission line theory (references 3, 13, and 16). The transmission line analogy solution is particularly useful for the direct establishment of the support conditions needed for optimum shaft operation through the critical speeds.

The purpose of this report is to present the development of a general analysis of the multisupported, supercritical shaft in terms of the transmission line analogy and to indicate the usefulness of these tools. The conditions of optimized supports corresponding to minimized dynamic responses will be established.

SCOPE

The basic mathematical concepts upon which the transmission line analogy proposed in this report is based were originally developed by Nelson in terms of a shaft on end supports only (reference 14). Liu, Friedericy, and Eppink extended this work to apply to shafts having one additional intermediate support (reference 8). The studies discussed in this report are concerned with an extension of Nelson's electrical transmission line analogy for supercritical shafts, to include the effects of any number of intermediate supports. The derivation of the equations of motion of supercritical shafts with respect to fixed reference axes incorporates the effects of rotational inertia, gyroscopics, and shear deformations.

Independent of the University of Virginia work, Voorhees and coworkers (reference 5) formulated their version of a transmission line analogy. In this version the fourth-order differential equation of motion for beams has been reduced to an approximating second-order equation which is completely analogous to the second-order equation which governs electrical network behavior. This reduction in order requires that a one-to-one relationship shall exist between moments and deflections of the shaft. Such

a relationship can only be brought about by the introduction of compromise boundary conditions at the supports. However, the approach has the advantage that all the terminology and computational aids developed for the electrical transmission line problem can be utilized directly in the design of supports for supercritical shafts. The support optimization formulas in the Design Manual for Supercritical-Speed Power-Transmission Shafts, prepared by the Battelle Memorial Institute for the U. S. Army Transportation Research Command and the U. S. Air Force Research and Technology Division (reference 2) are based on this direct approach, and they are extremely convenient for designing supercritical, power transmission shafting because of their simplicity and straightforwardness.

In the transmission analogy of this report, the fourth-order differential equation which governs supercritical shaft behavior is solved in an exact manner and the various component terms to the solution are worked into standing wave forms which are analogous to voltage wave forms. The optimization of support conditions is then performed in the same manner as loads are optimized in the electrical transmission line problem. Usually, the supports are formed with rotational and translational mass-spring-damper units and these are interpreted as impedances to the standing waves.

The development of simple formulas for the determination of optimum support values has been attempted; however, due to the complexity of the solution of the multisupported shaft, a shaft with only one interior support has been used as a specific illustration. The various formulations developed in the illustration should be useful in the design of optimum support conditions of supercritical shafts on three flexible supports and should complement the equations and results in the Design Manual mentioned earlier. The three support formulations of this report allow for the determination of optimum interior support parameters in terms of end supports which are not just fixed and simply supported, but may consist of rotational and/ or translational mass-spring-damper units. Specific results can be found in Chapter 4 of this report.

ASSUMPTIONS AND NOTATIONAL CONVENTIONS

It is shown in Figure 1 that the mathematical model of a prismatic shaft is embedded in a right-hand orthogonal normalized space coordinate system with fixed axes X_1, X_2, X_3. Let S_0 represent the shaft with elastic curve S. Let A_0 and B_0, where the end supports are attached, represent the end bodies with elastic curves A and B, respectively. A, S, and B coincide, at rest, with the intervals $[a, 0]$, $(0, \ell_n)$, and $[\ell_n, b]$, respectively, on the X_1-axis, where $a < 0 < \ell_n < b$. A closed interval and an open interval with end points α, β on the X_1-axis are represented by $[\alpha, \beta]$ and (α, β), respectively. Except for their common connection (the shaft, S_0), A_0 and B_0 are dynamically independent.

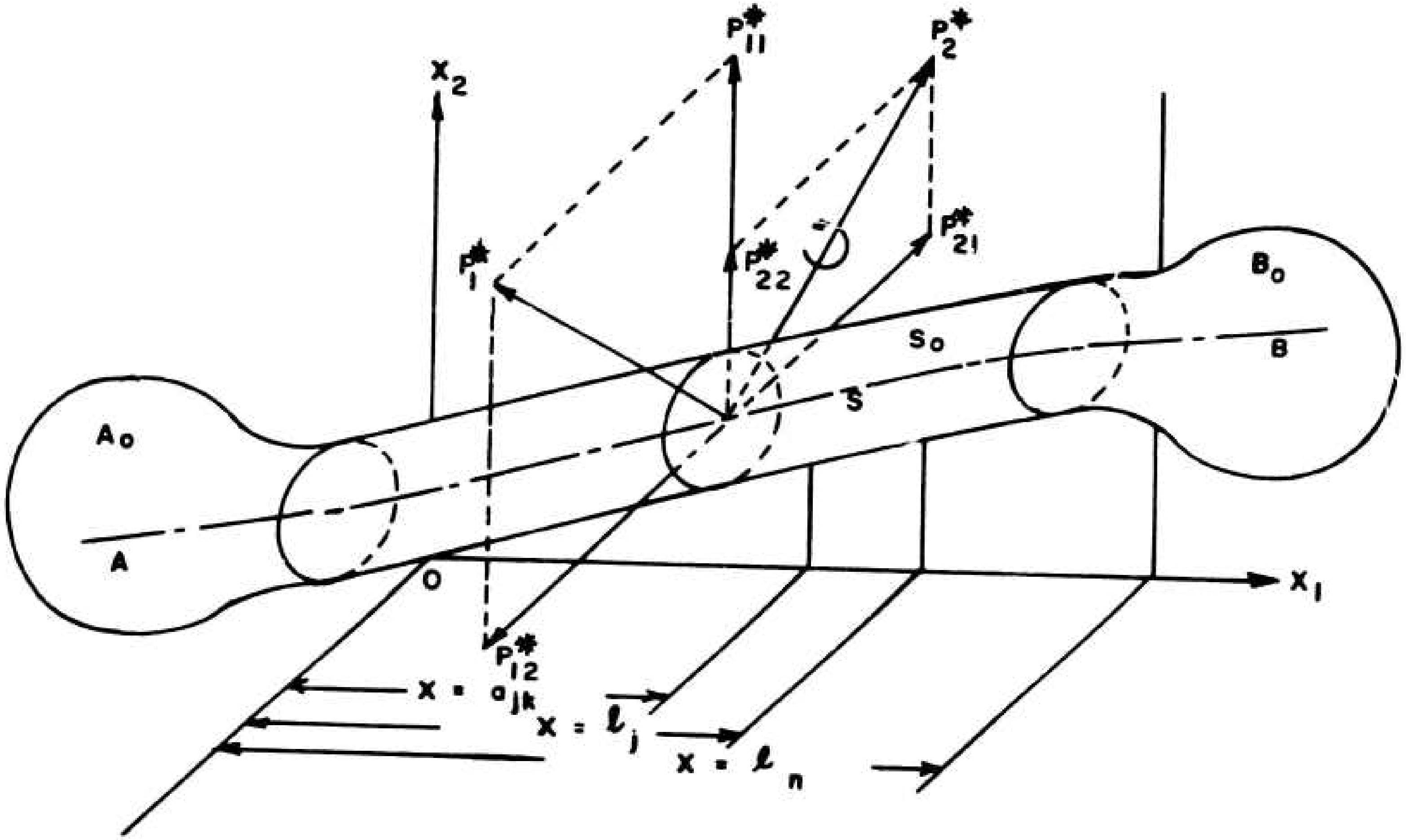

Figure 1. Pictorial Representation of A_o, S_o, B_o Embedded in X_1, X_2, X_3 With a Driving Force $P^*(a_{jk})$.

The intermediate supports are attached at $x = \ell_j$, $j = 1, 2, \ldots, n-1$, where $0 < \ell_1 < \ell_2 \ldots < \ell_{n-1} < \ell_n$. Force, torque, and motion along the X_1-axis are assumed to be zero. Constant angular velocity, small transverse motion, axial symmetry, and linearity are assumed throughout.

Where C denotes complex numbers, let C_n denote the n-dimensional vector space over a field of complex numbers with the elements, or vectors, of C_n being thought of as "column vectors". In the following, the appearance of a bar (possibly together with other symbols) over a quantity indicates that it is an element of C_n for some $n > 1$ (usually $n = 2$).

Let ASB denote collectively the bodies A_0, S_0, B_0. Let d(x) be that portion of ASB which, at rest, has X_1-coordinate x on [a, b].

A "mass element on [a, b]" is that portion of ASB which lies between $d(x - dx/2)$ and $d(x + dx/2)$ where dx is small. The normal to a differential mass element is the vector normal to d(x) at its intersection with the elastic curve, with positive orientation being the same as that of the X_1-axis. (See Figures 2 and 3.)

For points on elastic curves A, S, B, "position" as the vector is defined as follows:

$$\bar{Y}^* = \begin{bmatrix} Y_1^* \\ \\ Y_2^* \end{bmatrix} = \begin{bmatrix} Y_{11}^* + i\, Y_{12}^* \\ \\ Y_{21}^* + i\, Y_{22}^* \end{bmatrix} .$$

This is a continuous function of x and t to C_2, i.e., 2-dimensional vector space over a field of complex numbers, where x is on [a, b]. "*" denotes the unnormalized variables. Y_{11}^* and Y_{12}^* are the projections of the deflections at a general point of the elastic curves A, S, B, on the X_1X_2-, X_1X_3-planes, respectively. The normal to a differential mass element has projections on the planes X_1X_2, X_1X_3, which are inclined at acute angles with respect to the X_1-axis. These angles of inclination are Y_{21}^*, Y_{22}^*, respectively, where the positive senses of rotation are from X_2, X_3, respectively, to X_1. (See Figures 2 and 3.)

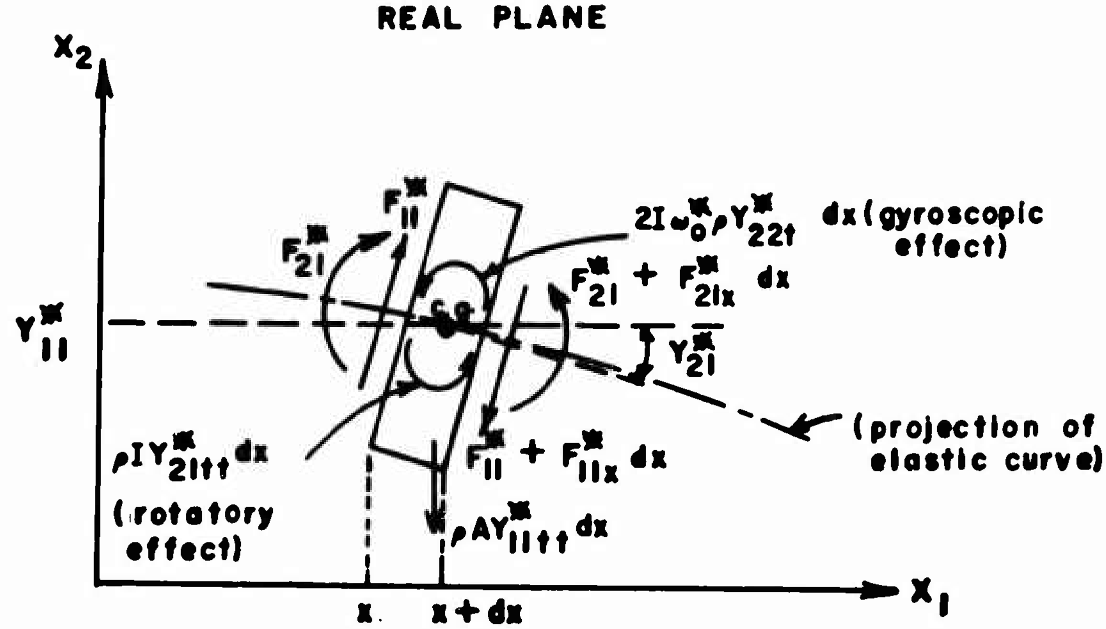

Figure 2. Projection of a Differential Shaft Element Upon X_1X_2 - Plane With Acting Forces.

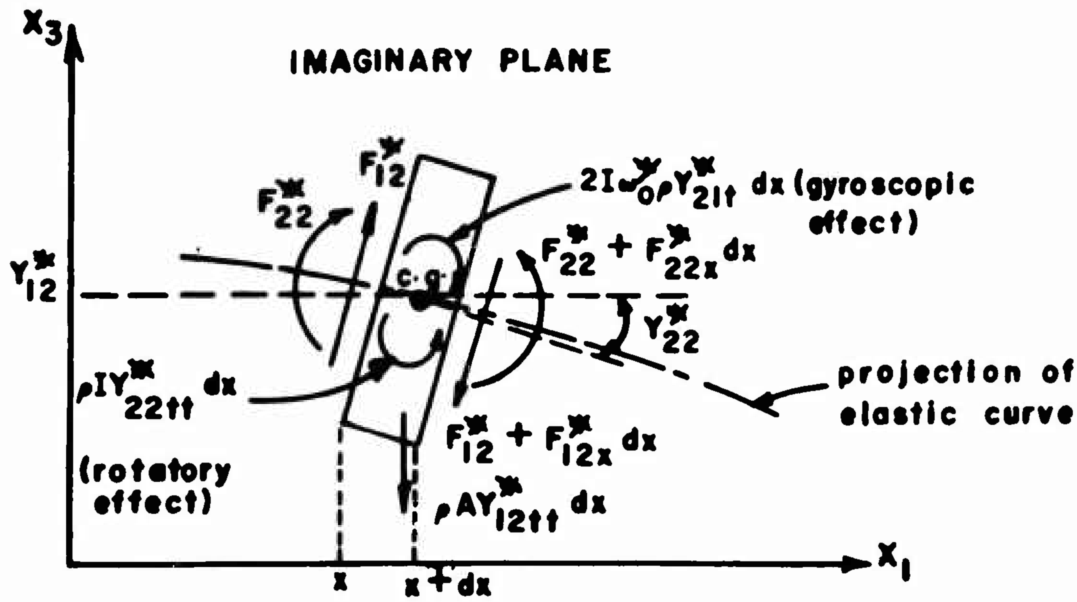

Figure 3. Projection of a Differential Shaft Element Upon X_1X_3 - Plane With Acting Forces.

Similarly, "force" as a vector is defined as follows:

$$\bar{Q}^* = \begin{bmatrix} Q_1^* \\ \\ Q_2^* \end{bmatrix} = \begin{bmatrix} Q_{11}^* + i\,Q_{12}^* \\ \\ Q_{21}^* + i\,Q_{22}^* \end{bmatrix} .$$

This is also a function of x and t to C_2, and x is on $[a, b]$. Q_{11}^* and Q_{12}^* are the components of force along the X_2-, X_3-axes, respectively; and Q_{21}^* and Q_{22}^* are the components of torque along the $-X_3$-, X_2-axes, respectively.

External forces on $(0, \ell_n)$ will be restricted to a series of concentrated driving forces which are caused by the mass eccentricities. The force vector

$$\bar{P}^* = \begin{bmatrix} P_1^* \\ \\ P_2^* \end{bmatrix} = \begin{bmatrix} P_{11}^* + i\,P_{12}^* \\ \\ P_{21}^* + i\,P_{22}^* \end{bmatrix}$$

denotes the "driving force", a function of x and t to C_2 where x is on $(0, \ell_n)$, which vanishes at a finite set of points. These points are located at $x = a_{jk}$; $j = 1, 2, \ldots; n$ and $k = 1, 2, \ldots; k(j)$, where $k(j)$ means that k may be a different integer for different spans. (See Figure 1.)

Similarly, "internal force" as a force vector is defined as follows:

$$\bar{F}^* = \begin{bmatrix} F_1^* \\ \\ F_2^* \end{bmatrix} = \begin{bmatrix} F_{11}^* + i\,F_{12}^* \\ \\ F_{21}^* + i\,F_{22}^* \end{bmatrix} .$$

This force vector is equal to the force applied to the differential mass element at $x - dx/2$. $\bar{F}$ is a piecewise continuous function of x and t to C_2, where x is on $[a, b]$, and the magnitude of each upward jump at a point of discontinuity equals the external concentrated force at that point. (See Figures 2 and 3.)

Throughout the following equations, a quantity with "#" above it is, unless otherwise specified, a 2×2 matrix with complex entries, and the inverse

of such a matrix will be denoted by attaching the superscript "-1". A tilde (perhaps together with other symbols) above a quantity denotes the Laplace transform of that quantity; e.g.,

$$\tilde{g}(x, s) = \int_0^\infty g(x, t)e^{-st}dt$$

where g is a function of x and t to C_2 where x is on $[a, b]$.

Assume throughout that $\bar{Y}(x, t) = \bar{0}$ and $d\bar{Y}(x, t)/dt = \bar{0}$ at $t = 0$ ($\bar{0}$ is the null vector in C_2). Physically, this means that the system is initially at rest or that the initial transverse deflection, angular deformation, transverse velocity and angular rotation are zero; the initial position of the shaft is coincident with the X_1-coordinate of the fixed reference frame. Thus, the solution will be limited to steady-state conditions by applying the Laplace transform technique to solve the governing differential equation, and the transient state is assumed to vanish automatically.

Let $\approx{Q}$ be an external force vector applied to $d(x)$ as defined before; let $\approx{Y}$ be the position vector of $d(x)$. Because of axial symmetry, the relation between the force vector, $\approx{Q}$, and the position vector, $\approx{Y}$, may be written in either of the two following forms:

$$\approx{Q} = s \overset{\#}{Z} \approx{Y}$$

or

$$\approx{Y} = \frac{1}{s} \overset{\#}{M} \approx{Q}$$

where "$\sim$" denotes the Laplace transformation variable. $\overset{\#}{Z}$ and $\overset{\#}{M}$ are called, respectively, impedance and mobility in 2×2 matrices with complex entries. If $\overset{\#}{Z}$ and $\overset{\#}{M}$ are non-singular, $\overset{\#}{Z}\,\overset{\#}{M} = \overset{\#}{M}\,\overset{\#}{Z} = \overset{\#}{I}$, where $\overset{\#}{I}$ is the 2×2 identity matrix. Impedance and mobility are functions of s, and their evaluation depends solely on the parameters of the dynamic system itself. Impedance is usually referred to as a transfer function in mathematics. It is assumed that the impedance or mobility at every support location is a known quantity or can be calculated.

CHAPTER 2

TRANSMISSION LINE ANALOGY SOLUTION OF THE SUPERCRITICAL SHAFT WITH ANY NUMBER OF INTERMEDIATE SUPPORTS

CONFIGURATIONS

A physical model of the shaft, which corresponds to the mathematical model in Figure 1, is presented in Figure 4. All the parameters used in the following theoretical analysis will be assumed to be positive in the sense indicated in the figure. The shaft end conditions and the conditions of all intermediate supports may be described as impedances or mobilities; however, for convenience, only impedances will be used throughout. The impedances of all supports are denoted as $\overset{\#}{Z}_0(0)$, $\overset{\#}{Z}_1(\ell_1)$, . . . , $\overset{\#}{Z}_n(\ell_n)$ where 0, ℓ_1, ℓ_2, . . . , ℓ_n indicate the locations of supports measured from the left end support. All the support impedances are assumed to be given.

$\tilde{\tilde{P}}(a_{jk})$ is the k^{th} driving force in the j^{th} span where j = 1, 2, . . . , n and k = 1, 2, . . . , k. The value of k may be different in different spans, depending on each span length and on how many differential mass elements are assumed to exist in each span. Hence, k will be written as k(j), which means that the value of k depends on the span number.

EQUATIONS OF MOTION FOR THE SHAFT

The equations which govern the behavior of rotating shafts include the effects of gyroscopic motion, rotational inertia, and shear deformations. The projections of an infinitesimal element of the shaft, of length dx and with all the forces acting on it, are shown in Figures 2 and 3. The symbols used in these two figures are listed as follows:

$*$	=	the unnormalized variable
ω_0^*	=	angular velocity of the rotating shaft
I	=	moment of inertia of the shaft cross-sectional area about a diameter
ρ	=	mass density of the shaft material
A	=	area of the shaft cross section.

Under certain conditions, such as at very high rotating speeds, not only the centrifugal forces of the infinitesimal rotating mass but also the

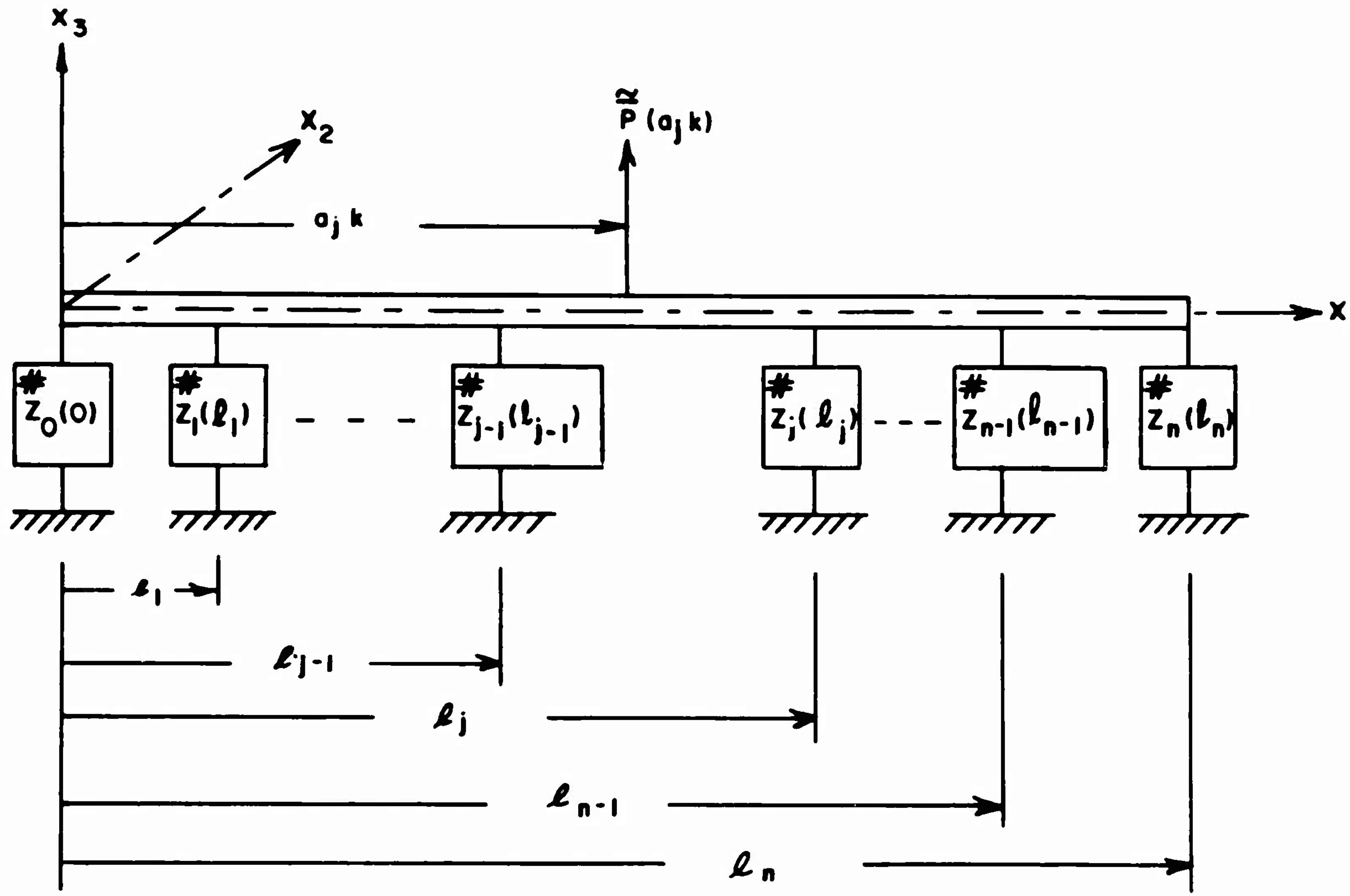

Figure 4. Uniform Shaft on Multiple Supports.

movements of the axes of the rotating mass are of importance and should be taken into account. The mathematical derivations of the expressions for the moments due to angular movements (rotatory and gyroscopic effects) and the expressions of inertia forces are included in Appendix A.

If the projection on the X_1X_2-plane, which is treated as a real plane, is considered, the following two equations can be obtained by applying equilibrium conditions and by neglecting higher order terms (see Figure 2):

By summing moments,

$$F^*_{11} - F^*_{21x} - \rho I Y^*_{21tt} - 2\rho I \omega^*_0 Y^*_{22t} = 0 \quad ; \tag{1}$$

By summing forces,

$$F^*_{11x} + \rho A Y^*_{11tt} = 0 \quad ; \tag{2}$$

By applying shear relationship,

$$F^*_{11} = K' A E_s(-Y^*_{11x} - Y^*_{21}) \quad ;$$

where

K' = numerical factor depending on the shape of cross section

= $\dfrac{\text{average shearing stress on cross section}}{\text{shearing stress at neutral axis}}$

= $\dfrac{Id}{AQ}$, where Q is the moment of cross-sectional area of shaft above the neutral axis with respect to this axis, and d is the thickness of section at the neutral axis, in this case the diameter of the circular section

E_s = shear modulus of shaft material

Y^*_{11x} = slope of deflection curve

Y^*_{21} = slope of deflection curve when shearing force is neglected

$-Y^*_{11x} - Y^*_{21}$ = change of slope deflection curve due to shearing force only.

The preceding equation may be rearranged as follows:

$$F^*_{11} + K' A E_s (Y^*_{11x} + Y^*_{21}) = 0 \quad . \tag{3}$$

By applying the bending relationship,

$$- F^*_{21} = \frac{Y^*_{21x}}{E_y I}$$

where

F^*_{21} = moment on the differential mass element

Y^*_{21x} = change of slope of deflection curve

E_y = Young's modulus of shaft material.

The above equation may be rearranged as

$$F^*_{21} + E_y I Y^*_{21x} = 0 . \tag{4}$$

Similarly, if the projection on the X_1X_3-plane, which is treated as an imaginary plane, is considered, and if the preceding arguments are used, the following four corresponding equations are obtained (see Figure 3):

$$F^*_{12} - F^*_{22x} - \rho I Y^*_{22tt} + 2\rho I \omega^*_0 Y^*_{21t} = 0$$

$$F^*_{12x} + \rho A \, Y^*_{12tt} = 0$$

$$F^*_{12} + K' A E_s (Y^*_{12x} + Y^*_{22}) = 0$$

$$F^*_{22} + E_y \, I \, Y^*_{22x} = 0.$$

If each pair of equations is considered, in complex variable form the corresponding four equations are obtained:

$$(F^*_{11} + i F^*_{12}) - (F^*_{21x} + iF^*_{22x}) - \rho I(Y^*_{21tt} + iY^*_{22tt}) + 2i\rho I \omega^*_0 (Y^*_{21t} + iY^*_{22t}) = 0$$

$$(F^*_{11x} + iF^*_{12x}) + \rho A (Y^*_{11tt} + iY^*_{12tt}) = 0$$

$$(F^*_{11} + iF^*_{12}) + K' AE_s \left[(Y^*_{11x} + iY^*_{12x}) + (Y^*_{21} + iY^*_{22}) \right] = 0$$

$$(F^*_{21} + iF^*_{22}) + E_y I (Y^*_{21x} + i Y^*_{22x}) = 0 \quad .$$

If the notations for position vector and force vector as defined on page 8 in Chapter 1 are applied, the equations may be written in more compact forms in referring to the framework of a 2-dimensional vector space over a field of complex numbers as opposed to the more general space of four dimensions over a field of real numbers.

$$F_1^* - F_{2x}^* - \rho I Y_{2tt}^* + 2i\rho I \omega_0^* Y_{2t}^* = 0$$

$$F_{1x}^* + \rho A Y_{1tt}^* = 0$$

$$F_1^* + K' A E_s (Y_{1x}^* + Y_2^*) = 0$$

$$F_2^* + E_y I Y_{2x}^* = 0 \quad .$$

For normalizing these equations of motion, R_b, $E_y A$, and R_b / c_s are used; they correspond to unit length, unit force, and unit time, respectively. R_b, c_s are, respectively, radius of gyration in bending and the velocity of sound for the shaft material. Note that $c_s = \sqrt{E_y / \rho}$. By applying the standard procedures for normalization, the differential equations of motion may be written as follows:

$$\left.\begin{aligned} & F_1 - F_{2x} - Y_{2tt} + 2i\omega_0 Y_{2t} = 0 \\ & F_{1x} + Y_{1tt} = 0 \\ & F_2 + Y_{2x} = 0 \\ & e' F_1 + Y_{1x} + Y_2 = 0 \end{aligned}\right\} \quad (5)$$

where $e' = (1/K')(E_y / E_s)$, which is a numerical coefficient depending on the shape of cross section and the shaft material, and where ω_0 is the angular velocity about X_1-axis, as before Y_1, Y_2 are, respectively, the transverse position and the inclination of normal of the elastic curve of shaft. Since all the initial values of the parameters appearing in Eqs. (5), i.e., the right-hand limits at t = 0, are null vectors, Eqs. (5) in the Laplace transform are as follows:

$$\left.\begin{aligned} & \tilde{F}_1 - \tilde{F}_{2x} + s(2i\omega_0 - s)\,\tilde{Y}_2 = 0 \\ & \tilde{F}_{1x} + s^2 \tilde{Y}_1 = 0 \\ & \tilde{F}_2 + \tilde{Y}_{2x} = 0 \\ & e' \tilde{F}_1 + \tilde{Y}_{1x} + \tilde{Y}_2 = 0 \quad . \end{aligned}\right\} \quad (6)$$

In Eqs. (5), if F_1, F_2, and Y_2 are eliminated, the governing differential equation in terms of Y_1 (transverse deflections) of the dynamic system may be obtained in the following form:

$$Y_{1xxxx} + Y_{1tt} - (1 + e')Y_{1xxtt} + e' Y_{1tttt} + 2i\omega_0(Y_{1xxt} - e' Y_{1ttt}) = 0 \quad . \qquad (7)$$

If, as has been noted, all the initial values for Y_1, Y_{1t}, Y_{1tt}, and Y_{1ttt} are zero, and if the Laplace transform is used again, Equation (7) may be transformed as follows:

$$\tilde{Y}_{1xxxx} + s\left[2i\omega_0 - s(1 + e')\right]\tilde{Y}_{1xx} + s^2\left[1 + se'(s - 2i\omega_0)\right]\tilde{Y}_1 = 0 \quad . \qquad (8)$$

The solution of Eq. (8) corresponds to the steady-state solution of Eq. (7), in which a solution is assumed in the form of

$$Y_1 = y(x)e^{i\omega t}$$

where ω is the frequency of the driving force and $y(x)$ is of the form

$$y(x) = Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x} + De^{m_4 x}$$

where m_1, m_2, m_3, and m_4 are roots of the characteristic equation of the governing differential Eq. (7). A, B, C, and D are arbitrary integration constants which may be determined by applying boundary conditions.

SOLUTIONS FOR THE EQUATIONS OF MOTION

With reference to Eq. (8), let

$$Y_1 = Ae^{m_1 x} + Be^{m_2 x} + Ce^{m_3 x} + De^{m_4 x} \quad .$$

Then, if the above expression is substituted into Eq. (8), the following characteristic equation is obtained:

$$m^4 + s\left[2i\omega_0 - s(1 + e')\right]m^2 + s\left[1 + se'(s - 2i\omega_0)\right] = 0$$

which yields

$$m_1, m_2 = \pm ie_1\sqrt{\omega}$$

where

ω = $-is$ or $s = i\omega$ (ω is the frequency response which is the same as the frequency of driving force, i.e., the angular velocity ω_0 of the rotating shaft itself)

$$e_1 = \sqrt{1/e_3 + e'\omega}$$

$$e_3 = \sqrt{1 + \alpha^2} + \alpha$$

$$\alpha = \omega_0 + \frac{1}{2}\omega(e' - 1)$$

and

$$m_3, m_4 = \pm e_2\sqrt{\omega}$$

where

$$e_2 = \sqrt{e_3 - e'\omega} .$$

The replacement of s by $i\omega$ corresponds to the steady-state solution. If the relationships in Eqs. (6) are applied, the following general solutions for $\tilde{Y}_2$, $\tilde{F}_1$, $\tilde{F}_2$ are obtained:

$$\tilde{Y}_1 = Ae^{m_1x} + Be^{m_2x} + Ce^{m_3x} + De^{m_4x}$$

$$\tilde{Y}_2 = \omega^{1/2}\left[-\frac{i}{e_1e_3}Ae^{m_1x} + \frac{i}{e_1e_3}Be^{m_2x} - \frac{e_3}{e_2}Ce^{m_3x} + \frac{e_3}{e_2}De^{m_4x}\right]$$

$$\tilde{F}_1 = \omega^{3/2}\left[-\frac{i}{e_1}Ae^{m_1x} + \frac{i}{e_1}Be^{m_2x} + \frac{1}{e_2}Ce^{m_3x} - \frac{1}{e_2}De^{m_4x}\right]$$

$$\tilde{F}_2 = \omega\left[-\frac{1}{e_3}Ae^{m_1x} - \frac{1}{e_3}Be^{m_2x} + e_3Ce^{m_3x} + e_3De^{m_4x}\right].$$

If the following set of arbitrary constants (q_1, q_2, r_1, r_2) is introduced:

$$A = -\frac{i}{s}\omega^{p+1}e_1e_3r_1$$

$$B = -\frac{i}{s}\omega^{p+1}e_1e_3q_1$$

$$C = -\frac{i}{s}\omega^{p+1}e_2r_2$$

$$D = -\frac{i}{s}\omega^{p+1}e_2q_2$$

where p is an arbitrary real number which, once chosen, remains the same, and where q_1, q_2, r_1, r_2 are complex numbers; therefore, the expressions for Y_1, Y_2, F_1, F_2 can be rewritten as follows:

$$\left.\begin{aligned}
\tilde{Y}_1 &= \frac{1}{s}\omega^p\Big[-e_1e_3\omega e^{-ie_1\sqrt{\omega}x}q_1 - ie_2\omega\, e^{-e_2\sqrt{\omega}\,x}q_2 - \\
&\qquad ie_1e_3\omega e^{ie_1\sqrt{\omega}x}r_1 - ie_2\omega e^{e_2\sqrt{\omega}\,x}r_2\Big] \\
\tilde{Y}_2 &= \frac{1}{s}\omega^p\Big[\omega^{3/2}e^{-ie_1\sqrt{\omega}\,x}q_1 - ie_3\omega^{3/2}e^{-e_2\sqrt{\omega}\,x}q_2 - \\
&\qquad \omega^{3/2}e^{ie_1\sqrt{\omega}x}r_1 + ie_3\omega^{3/2}e^{e_2\sqrt{\omega}x}r_2\Big] \\
\tilde{F}_1 &= \omega^p\Big[-ie_3\omega^{3/2}e^{-ie_1\sqrt{\omega}x}q_1 + \omega^{3/2}e^{-e_2\sqrt{\omega}\,x}q_r + \\
&\qquad ie_3\omega^{3/2}e^{ie_1\sqrt{\omega}\,x}r_1 - \omega^{3/2}e^{e_2\sqrt{\omega}\,x}r_2\Big] \\
\tilde{F}_2 &= \omega^p\Big[e_1\omega\, e^{-ie_1\sqrt{\omega}\,x}q_1 - e_2e_3\omega\, e^{-e_2\sqrt{\omega}\,x}q_2 + \\
&\qquad e_1\omega\, e^{ie_1\sqrt{\omega}\,x}r_1 - e_2e_3\omega\, e^{e_2\sqrt{\omega}\,x}r_2\Big]
\end{aligned}\right\}\quad(9)$$

By defining

$$\bar{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \text{and} \qquad \bar{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$$

as two arbitrary constant vectors in a 2-dimensional vector space over a field of complex numbers, and by recalling the expressions of the position and the force vectors as defined under "Assumptions and Notational Conventions" in Chapter 1, Eqs. (9) may be rewritten in a much more compact form, as follows:

$$\begin{bmatrix} \tilde{\tilde{Y}} \\ \\ \tilde{\tilde{F}} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} \overset{\#}{C}_y & \overset{\#}{0} \\ \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x) & \overset{\#}{0} \\ \\ \overset{\#}{0} & \overset{\#}{R}(-x) \end{bmatrix} \begin{bmatrix} \bar{q} \\ \\ \ddot{r} \end{bmatrix} \tag{10}$$

where $\overset{\#}{C}_y$, $\overset{\#}{C}_f$, $\overset{\#}{C}_{11}$, $\overset{\#}{C}_{12}$, $\overset{\#}{C}_{21}$, and $\overset{\#}{C}_{22}$ are as shown in Appendix B, and $\overset{\#}{0}$ is the 2×2 null matrix.

The "propagation matrix", $\overset{\#}{R}(\alpha)$, is a function on the real numbers defined by

$$\overset{\#}{R}(\alpha) = \begin{bmatrix} \text{cis}\,(e_1\sqrt{\omega}\,\alpha) & 0 \\ \\ 0 & \exp(-e_2\sqrt{\omega}\,\alpha) \end{bmatrix}$$

where

$$\text{cis}\,\theta = e^{i\theta} = \cos\theta + i\sin\theta$$

$$\exp\theta = e^{\theta}$$

$$\overset{\#}{R}(0) = \overset{\#}{I} \quad .$$

Simply by expanding the matrix form of Eq. (10), this equation may be verified as being identical to Eqs. (9).

BOUNDARY CONDITIONS

By applying the boundary conditions at $x = 0$, ℓ_1, ℓ_2, . . . , ℓ_n, and a_{jk} on the X_1-axis, the integration constants appearing in the general solution of the governing differential Eq. (8) can be determined. (See Eq. 10.)

At Left End Support

The boundary condition at $x = 0-0$, i.e., the left end support, may be obtained by considering the impedance at this point. (See Figure 5.)

Since $\tilde{\tilde{F}}$ at $x = 0-0$ is in the negative direction, according to the sign convention defined in Chapter 1, the boundary condition of the left end is expressed as:

$$-\tilde{\tilde{F}}(0) = s\,\overset{\#}{Z}_0(0)\,\tilde{\tilde{Y}}(0) \tag{11}$$

which should be satisfied at $x = 0-0$; -0 denotes the left-hand limit.

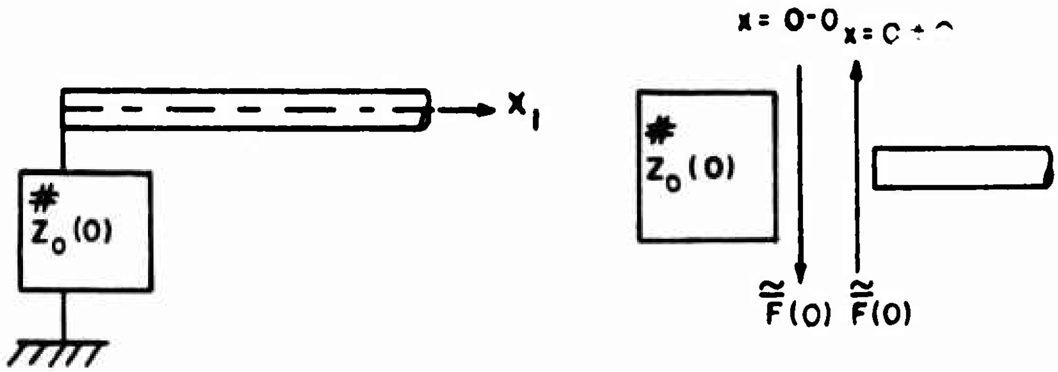

Figure 5. Left End Support.

<u>At Right End Support</u>

The boundary condition at $x = \ell_n + 0$, i.e., the right end support, may be obtained in a similar way, as follows:

$$\overset{\approx}{F}(\ell_n) = s\,\overset{\#}{Z}_n(\ell_n)\,\overset{\approx}{Y}(\ell_n). \tag{12}$$

Note that $\overset{\approx}{F}$ is in the positive direction at $x = \ell_n + 0$; also, $+0$ denotes the right-hand limit. Eq. (12) is the condition that should be satisfied at $x = \ell_n + 0$. (See Figure 6.)

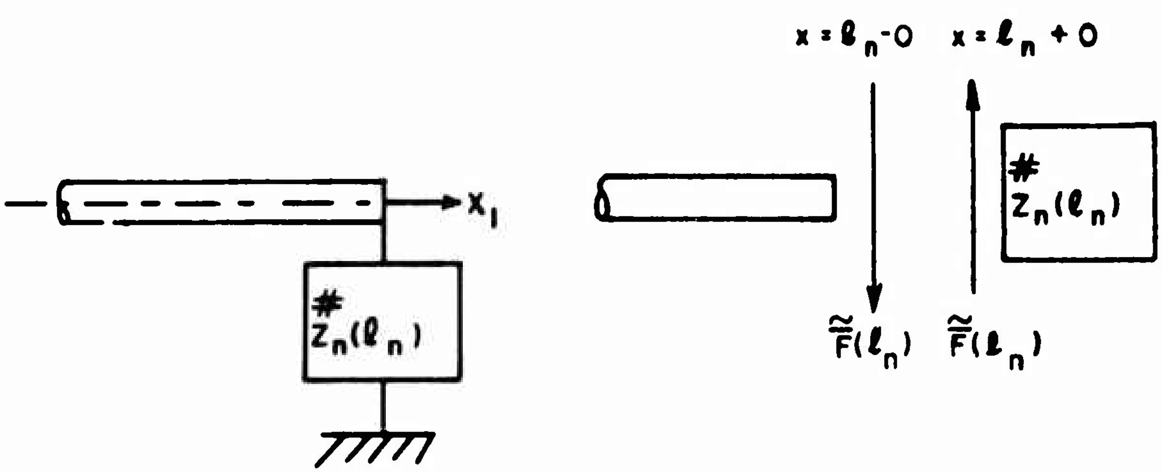

Figure 6. Right End Support.

At Intermediate Supports

The boundary conditions at $x = \ell_j$, $j = 1, 2, \ldots, n-1$, may be obtained

by considering the continuity of the deflection curve and the force equilibrium at every intermediate support. (See Figure 7.)

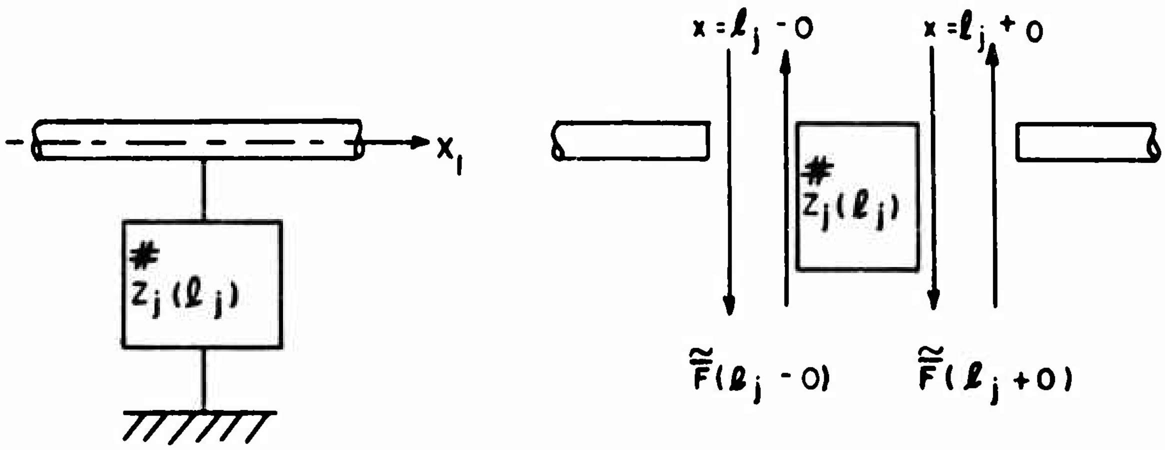

Figure 7. Intermediate Supports.

If the impedance is isolated at $x = \ell_j$ and if the net force applied at $x = \ell_j$ is $\left[\tilde{\tilde{F}}(\ell_j-0) - \tilde{\tilde{F}}(\ell_j+0)\right]$ and if the deflections at $x = \ell_j-0$ and at $x = \ell_j+0$ are the same, the boundary conditions at $x = \ell_j$ are as follows:

$$\begin{aligned} \tilde{\tilde{Y}}(\ell_j-0) &= \tilde{\tilde{Y}}(\ell_j+0) = \tilde{\tilde{Y}}(\ell_j) \\ \tilde{\tilde{F}}(\ell_j-0) - \tilde{\tilde{F}}(\ell_j+0) &= s\,\overset{\#}{Z}_j(\ell_j)\tilde{\tilde{Y}}(\ell_j) \end{aligned} \tag{13}$$

$$j = 1, 2, \ldots, n-1$$

where +0 and -0 denote right-hand and left-hand limit, respectively.

At Driving Forces

The boundary conditions at the locations where driving forces are being applied may be obtained in a similar manner. If it is assumed that a

single concentrated driving force, $\overset{\approx}{\overline{P}}(a_{jk})$ is applied at $x = a_{jk}$ (i.e., the k^{th} driving force in the j^{th} span), the corresponding boundary conditions at $x = a_{jk}$ are as follows (see Figure 8):

$$\begin{aligned} \overset{\approx}{\overline{Y}}(a_{jk}-0) &= \overset{\approx}{\overline{Y}}(a_{jk}+0) = \overset{\approx}{\overline{Y}}(a_{jk}) \\ \overset{\approx}{\overline{F}}(a_{jk}-0) &+ \overset{\approx}{\overline{P}}(a_{jk}) - \overset{\approx}{\overline{F}}(a_{jk}+0) = 0 \quad . \end{aligned} \tag{14}$$

REFLECTION MATRICES AT SUPPORTS IN TERMS OF IMPEDANCES

If all the boundary conditions except those at $x = a_{jk}$ are applied, as mentioned in the preceding section, some useful functions can be found during the process of evaluating boundary conditions. (See the complete mathematical derivations in Appendix C.)

At $x = 0$ on the X_1-axis, the reflection matrix (looking to the left at $x = 0$) is defined in terms of end impedance by

$$\overset{\#}{\Gamma}_0(0) = \left[- \overset{\#}{C}_{22} + \overset{\triangle}{z}_0(0)\overset{\#}{C}_{12} \right]^{-1} \left[\overset{\#}{C}_{21} - \overset{\triangle}{z}_0(0)\overset{\#}{C}_{11} \right] \tag{15}$$

where

$$\overset{\triangle}{Z} = E \overset{\#}{Z} E, \qquad E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} ,$$

$$\overset{\#}{z} = \frac{1}{\sqrt{\omega}} F \overset{\#}{Z} F, \qquad F = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\omega} \end{bmatrix} .$$

At $x = \ell_1, \ell_2, \ldots, \ell_{j-1}, \ldots, \ell_{n-1}$, the reflection matrices looking to the left are defined as follows:

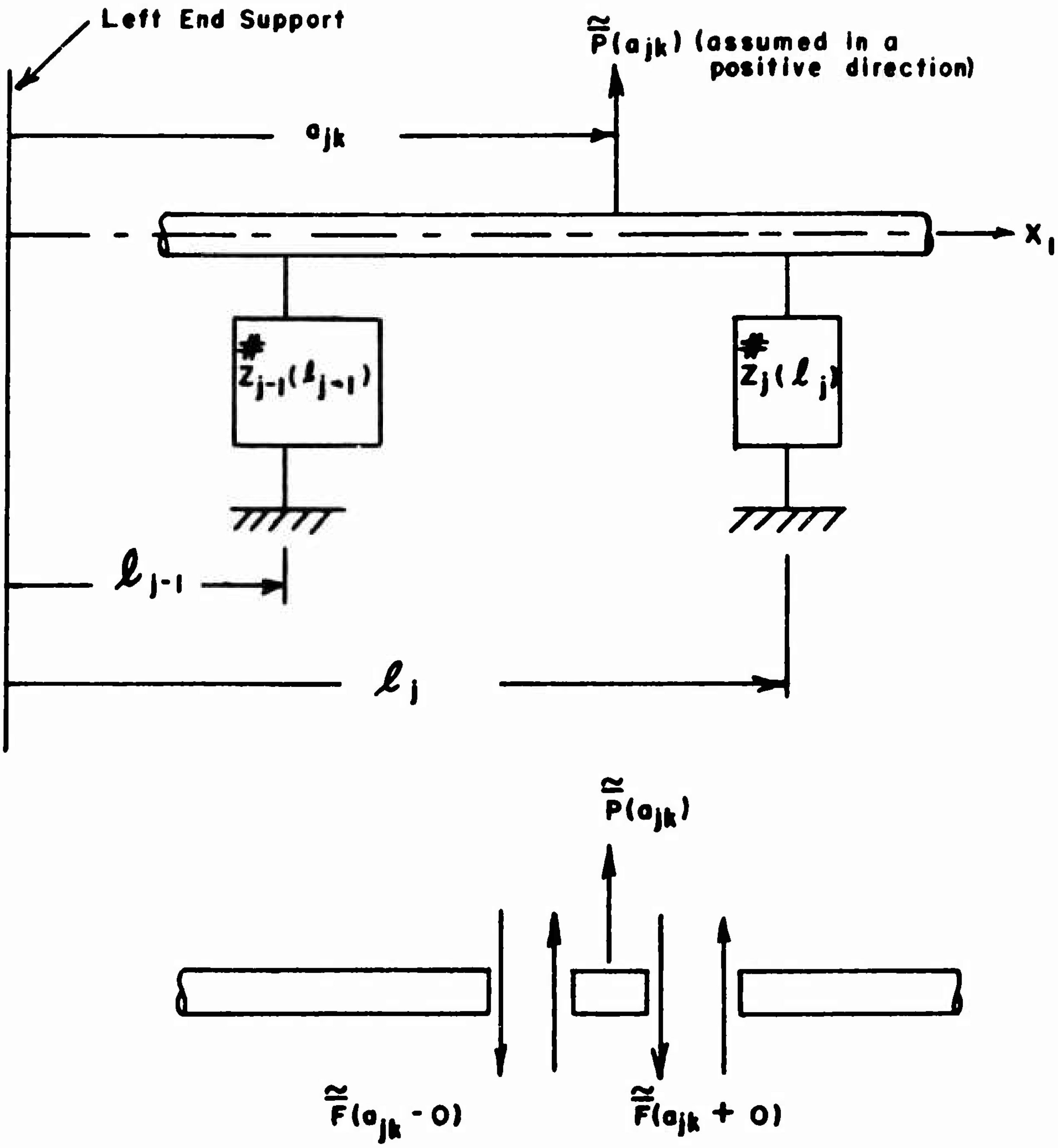

Figure 8. Forces at $X = a_{jk}$.

$$
\left.
\begin{aligned}
\overset{\#}{\Gamma}_{1\ell}(\ell_1) &= \left[-\overset{\#}{C}_{22}+\overset{\triangle}{z}_{1\ell}(\ell_1)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\triangle}{z}_{1\ell}(\ell_1)\overset{\#}{C}_{11}\right] \\
\overset{\#}{\Gamma}_{2\ell}(\ell_2) &= \left[-\overset{\#}{C}_{22}+\overset{\triangle}{z}_{2\ell}(\ell_2)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\triangle}{z}_{2\ell}(\ell_2)\overset{\#}{C}_{11}\right] \\
&\;\;\vdots \\
\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1}) &= \left[-\overset{\#}{C}_{22}+\overset{\triangle}{z}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\triangle}{z}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{C}_{11}\right] \\
&\;\;\vdots \\
\overset{\#}{\Gamma}_{(n-1)\ell}(\ell_{n-1}) &= \left[-\overset{\#}{C}_{22}+\overset{\triangle}{z}_{(n-1)\ell}(\ell_{n-1})\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\triangle}{z}_{(n-1)\ell}(\ell_{n-1})\overset{\#}{C}_{11}\right]
\end{aligned}
\right\} \quad (16)
$$

where subscripts $1, 2, \ldots, j-1, \ldots, n-1$ indicate the locations of supports; subscript ℓ indicates that the reflection matrices are being looked at to the left of $x = \ell_1, \ell_2, \ldots, \ell_{j-1}, \ldots, \ell_{n-1}$, $\overset{\triangle}{z}_{1\ell}(\ell_1), \overset{\triangle}{z}_{2\ell}(\ell_2), \ldots, \overset{\triangle}{z}_{(j-1)\ell}(\ell_{j-1}), \ldots, \overset{\triangle}{z}_{(n-1)\ell}(\ell_{n-1})$, represent, respectively, the total impedance looking to the left at $x = \ell_1, \ell_2, \ldots, \ell_1, \ell_2, \ldots, \ell_{j-1}, \ldots, \ell_{n-1}$, which are defined as follows:

$$\left.\begin{aligned}
\overset{\Delta}{z}_{1\ell}(\ell_1) &= \overset{\Delta}{z}_1(\ell_1) + \overset{\Delta}{z}_{0\ell}(\ell_1) \\
\overset{\Delta}{z}_{2\ell}(\ell_2) &= \overset{\Delta}{z}_2(\ell_2) + \overset{\Delta}{z}_{1\ell}(\ell_2) \\
&\;\;\vdots \qquad\qquad \vdots \\
\overset{\Delta}{z}_{(j-1)\ell}(\ell_{j-1}) &= \overset{\Delta}{z}_{j-1}(\ell_{j-1}) + \overset{\Delta}{z}_{(j-2)\ell}(\ell_{j-1}) \\
&\;\;\vdots \qquad\qquad \vdots \\
\overset{\Delta}{z}_{(n-1)\ell}(\ell_{n-1}) &= \overset{\Delta}{z}_{n-1}(\ell_{n-1}) + \overset{\Delta}{z}_{(n-2)\ell}(\ell_{n-1})
\end{aligned}\right\} \tag{17}$$

where the terms $\overset{\Delta}{z}_{0\ell}(\ell_1)$, $\overset{\Delta}{z}_{1\ell}(\ell_2)$, . . . , $\overset{\Delta}{z}_{(j-2)\ell}(\ell_{j-1})$, . . . , $\overset{\Delta}{z}_{(n-2)\ell}(\ell_{n-1})$ will be explained in the next section.

At $x = \ell_n$ on the X_1-axis, the reflection matrix (looking to the right) is defined in terms of the end impedance by

$$\overset{\#}{\Gamma}_n(\ell_n) = \left[-\overset{\#}{C}_{22} + \overset{\#}{z}_n(\ell_n)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21} - \overset{\#}{z}_n(\ell_n)\overset{\#}{C}_{11}\right] . \tag{18}$$

At $x = \ell_{n-1}, \ell_{n-2}, \ldots, \ell_j, \ldots, \ell_1$, the reflection matrices looking to the right are defined as follows:

$$\left.\begin{aligned}
\overset{\#}{\Gamma}_{(n-1)r}(\ell_{n-1}) &= \left[-\overset{\#}{C}_{22}+\overset{\#}{z}_{(n-1)r}(\ell_{n-1})\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\#}{z}_{(n-1)r}(\ell_{n-1})\overset{\#}{C}_{11}\right]\\
\overset{\#}{\Gamma}_{(n-2)r}(\ell_{n-2}) &= \left[-\overset{\#}{C}_{22}+\overset{\#}{z}_{(n-2)r}(\ell_{n-2})\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\#}{z}_{(n-2)r}(\ell_{n-2})\overset{\#}{C}_{11}\right]\\
&\ \ \vdots \\
\overset{\#}{\Gamma}_{jr}(\ell_{j}) &= \left[-\overset{\#}{C}_{22}+\overset{\#}{z}_{jr}(\ell_{j})\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\#}{z}_{jr}(\ell_{j})\overset{\#}{C}_{11}\right]\\
&\ \ \vdots \\
\overset{\#}{\Gamma}_{1r}(\ell_{1}) &= \left[-\overset{\#}{C}_{22}+\overset{\#}{z}_{1r}(\ell_{1})\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\#}{z}_{1r}(\ell_{1})\overset{\#}{C}_{11}\right]
\end{aligned}\right\}(19)$$

where subscript r indicates that the reflection matrix is being looked at to the right of $x = \ell_{n-1}, \ell_{n-2}, \ldots, \ell_j, \ldots, \ell_1$.
$\overset{\#}{z}_{(n-1)r}(\ell_{n-1}), \overset{\#}{z}_{(n-2)r}(\ell_{n-2}), \ldots, \overset{\#}{z}_{jr}(\ell_j), \ldots, \overset{\#}{z}_{1r}(\ell_1)$ represent, respectively, the total impedance looking to the right of $x = \ell_{n-1}$, $\ell_{n-2}, \ldots, \ell_j, \ldots, \ell_1$, which are defined as follows:

$$\left.\begin{aligned}
\overset{\#}{z}_{(n-1)r}(\ell_{n-1}) &= \overset{\#}{z}_{n-1}(\ell_{n-1}) + \overset{\#}{z}_{nr}(\ell_{n-1})\\
\overset{\#}{z}_{(n-2)r}(\ell_{n-2}) &= \overset{\#}{z}_{n-2}(\ell_{n-2}) + \overset{\#}{z}_{(n-1)r}(\ell_{n-2})\\
&\ \ \vdots
\end{aligned}\right\}(20)$$

$$
\left.
\begin{aligned}
\overset{\#}{z}_{jr}(\ell_j) &= \overset{\#}{z}_j(\ell_j) + \overset{\#}{z}_{(j+1)r}(\ell_j) \\
&\ \ \vdots \qquad\qquad \vdots \\
\overset{\#}{z}_{1r}(\ell_1) &= \overset{\#}{z}_1(\ell_1) + \overset{\#}{z}_{2r}(\ell_1)
\end{aligned}
\right\}
$$

where the terms $\overset{\#}{z}_{nr}(\ell_{n-1})$, $\overset{\#}{z}_{(n-1)r}(\ell_{n-2})$, . . . , $\overset{\#}{z}_{(j+1)r}(\ell_j)$, . . . , $\overset{\#}{z}_{2r}(\ell_1)$ will be explained in the next section.

For each closed interval $\left[0, \ell_1\right]$, $\left[\ell_1, \ell_2\right]$, . . . , $\left[\ell_{j-1}, \ell_j\right]$, . . . , $\left[\ell_{n-1}, \ell_n\right]$ on the X_1-axis, the corresponding reflection matrices looking to the left are, respectively, defined as follows:

$$
\left.
\begin{aligned}
\overset{\#}{\Gamma}_0(x) &= \overset{\#}{R}(x)\overset{\#}{\Gamma}_0(0)\overset{\#}{R}(x) \\
\overset{\#}{\Gamma}_{1\ell}(x) &= \overset{\#}{R}(x-\ell_1)\overset{\#}{\Gamma}_{1\ell}(\ell_1)\overset{\#}{R}(x-\ell_1) \\
&\ \ \vdots \qquad\qquad \vdots \\
\overset{\#}{\Gamma}_{(j-1)\ell}(x) &= \overset{\#}{R}(x-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(x-\ell_{j-1}) \\
&\ \ \vdots \qquad\qquad \vdots \\
\overset{\#}{\Gamma}_{(n-1)\ell}(x) &= \overset{\#}{R}(x-\ell_{n-1})\overset{\#}{\Gamma}_{(n-1)\ell}(\ell_{n-1})\overset{\#}{R}(x-\ell_{n-1}) \ .
\end{aligned}
\right\} \qquad (21)
$$

For each closed interval $\left[\ell_{n-1}, \ell_n\right]$, $\left[\ell_{n-2}, \ell_{n-1}\right]$, . . . , $\left[\ell_{j-1}, \ell_j\right]$, . . . , $\left[0, \ell_1\right]$ on the X_1-axis, the corresponding reflection matrices

looking to the right are, respectively, defined as follows:

$$\left.\begin{aligned}
\overset{\#}{\Gamma}_{n}(x) &= \overset{\#}{R}(\ell_n - x)\overset{\#}{\Gamma}_{n}(\ell_n)\overset{\#}{R}(\ell_n - x) \\
\overset{\#}{\Gamma}_{(n-1)r}(x) &= \overset{\#}{R}(\ell_{n-1} - x)\overset{\#}{\Gamma}_{(n-1)r}(\ell_{n-1})\overset{\#}{R}(\ell_{n-1} - x) \\
&\quad\vdots \qquad\qquad\qquad \vdots \\
\overset{\#}{\Gamma}_{jr}(x) &= \overset{\#}{R}(\ell_{j-x})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j - x) \\
&\quad\vdots \qquad\qquad\qquad \vdots \\
\overset{\#}{\Gamma}_{1r}(x) &= \overset{\#}{R}(\ell_1 - x)\overset{\#}{\Gamma}_{1r}(\ell_1)\overset{\#}{R}(\ell_1 - x) \quad .
\end{aligned}\right\} \qquad (22)$$

IMPEDANCES IN TERMS OF REFLECTION MATRICES AND SHAFT CHARACTERISTIC IMPEDANCE

It is shown in Appendix D that $\overset{\Delta}{z}_{0\ell}(x)$, the total impedance looking to the left of a generic point x where x is on $\left[0,\ \ell_1\right]$, is related to $\overset{\#}{\Gamma}_0(x)$ by the equation

$$\text{On } \left[0,\ \ell_1\right],\ \overset{\Delta}{z}_{0\ell}(x) = \left[\overset{\#}{C}_{21} + \overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(x)\right]\left[\overset{\#}{C}_{11} + \overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(x)\right]^{-1}$$

Similarly,

$$\left.\begin{aligned}
&\text{On } \left[\ell_1, \ell_2\right],\ \overset{\Delta}{z}_{1\ell}(x) = \left[\overset{\#}{C}_{21} + \overset{\#}{C}_{22}\overset{\#}{\Gamma}_{1\ell}(x)\right]\left[\overset{\#}{C}_{11} + \overset{\#}{C}_{12}\overset{\#}{\Gamma}_{1\ell}(x)\right]^{-1} \\
&\qquad\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}\right\} \qquad (23)$$

$$\left.\begin{aligned}
&\text{On}\left[\ell_{j-1},\ \ell_j\right],\quad \overset{\Delta}{z}_{(j-1)\ell}(x)=\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{(j-1)\ell}(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-1)\ell}(x)\right]^{-1}\\
&\quad\vdots\qquad\qquad\qquad\qquad\qquad\qquad\vdots\\
&\text{On}\left[\ell_{n-1},\ \ell_n\right],\quad \overset{\Delta}{z}_{(n-1)\ell}(x)=\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{(n-1)\ell}(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(n-1)\ell}(x)\right]^{-1}.
\end{aligned}\right\}$$

It is also shown in Appendix D that the other similar set of equations which represent the total impedance at x looking to the right is:

$$\left.\begin{aligned}
&\text{On}\left[\ell_{n-1},\ \ell_n\right],\quad \overset{\#}{z}_{nr}(x)=\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{n}(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{n}(x)\right]^{-1}\\
&\text{On}\left[\ell_{n-2},\ \ell_{n-1}\right],\quad \overset{\#}{z}_{(n-1)r}(x)=\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{(n-1)r}(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(n-1)r}(x)\right]^{-1}\\
&\quad\vdots\qquad\qquad\qquad\qquad\qquad\qquad\vdots\\
&\text{On}\left[\ell_{j-1},\ \ell_j\right],\quad \overset{\#}{z}_{jr}(x)=\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{jr}(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{jr}(x)\right]^{-1}\\
&\quad\vdots\qquad\qquad\qquad\qquad\qquad\qquad\vdots\\
&\text{On}\left[0,\ \ell_1\right],\quad \overset{\#}{z}_{1r}(x)=\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{1r}(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{1r}(x)\right]^{-1}.
\end{aligned}\right\}\quad(24)$$

The expressions for $\overset{\Delta}{z}_{0\ell}(\ell_1)$, $\overset{\Delta}{z}_{1\ell}(\ell_2)$, . . . , $\overset{\Delta}{z}_{(j-2)\ell}(\ell_{j-1})$, . . . , $\overset{\Delta}{z}_{(n-2)\ell}(\ell_{n-1})$ in Eqs. (17), and the expressions for $\overset{\#}{z}_{nr}(\ell_{n-1})$, $\overset{\#}{z}_{(n-1)r}(\ell_{n-2})$, . . . , $\overset{\#}{z}_{(j+1)r}(\ell_j)$, . . . , $\overset{\#}{z}_{2r}(\ell_1)$ in Eqs. (20) can be obtained from Eqs. (23) and Eqs. (24), respectively. Eqs. (23) and (24) will also enable us to evaluate the impedance looking both to the left and to the right at any location along the shaft.

Let

$$\overset{\#}{z}_s = \overset{\#}{C}_{21}\,\overset{\#}{C}{}_{11}^{-1} \tag{25}$$

which is expanded in complete matrix form in Appendix B, and

$$\overset{\#}{z}_s = \frac{1}{\sqrt{\omega}}\, F\,\overset{\#}{Z}_s\,F, \qquad F = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\omega} \end{bmatrix}$$

where $\overset{\#}{Z}_s$ is called the characteristic impedance of the rotating shaft. Note that $\overset{\#}{Z}_s$ is a function of frequency. The significance of the characteristic impedance will be discussed in Chapter 3.

COMPLETE SOLUTION IN MATRIX FORM

If all the boundary conditions beginning on page 17 of this chapter are used, all the integration constants appearing in the general solution can be determined. (See the complete mathematical derivations in Appendix E.) The complete solution, from which one may calculate the dynamic response, i.e., deflections and internal forces caused by the single driving force $\overset{\simeq}{P}(a_{jk})$, at any location along the shaft, may be written in the following matrix form:

$$\left.\begin{aligned}
&\text{On } \left[0,\ \ell_1\right], \\
&\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\overset{\#}{\Gamma}_0(0)\,\bar{r}_1 \\ \overset{\#}{R}(-x)\bar{r}_1 \end{bmatrix} \\
&\text{On } \left[\ell_1,\ \ell_2\right], \\
&\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\overset{\#}{\Gamma}_{1\ell}(0)\,\bar{r}_2 \\ \overset{\#}{R}(-x)\,r_2 \end{bmatrix}
\end{aligned}\right| \tag{26}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

On $\left[\ell_{j-2}, \ell_{j-1}\right]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-2)\ell}(0)\,\bar{r}_{j-1} \\ \overset{\#}{R}(-x)\bar{r}_{j-1} \end{bmatrix}$$

On $\left[\ell_{j-1}, a_{jk}\right]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\,\bar{r}_{j\ell} \\ \overset{\#}{R}(-x)\,\bar{r}_{j\ell} \end{bmatrix}$$

On $\left[a_{jk}, \ell_j\right]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\, q_{jr} \\ \overset{\#}{R}(-x)\overset{\#}{\Gamma}_{jr}(0)\,\bar{q}_{jr} \end{bmatrix}$$

On $\left[\ell_j, \ell_{j+1}\right]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\, \bar{q}_{j+1} \\ \overset{\#}{R}(-x)\overset{\#}{\Gamma}_{(j+1)r}(0)\, \bar{q}_{j+1} \end{bmatrix}$$

$$\vdots \qquad\qquad \vdots \tag{26}$$

On $\left[\ell_{n-2}, \ell_{n-1}\right]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\, \bar{q}_{n-1} \\ \overset{\#}{R}(-x)\overset{\#}{\Gamma}_{(n-1)r}(0)\, \bar{q}_{n-1} \end{bmatrix}$$

On $\left[\ell_{n-1}, \ell_n\right]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\, \bar{q}_{n} \\ \overset{\#}{R}(-x)\overset{\#}{\Gamma}_{n}(0)\, \bar{q}_{n} \end{bmatrix}$$

where

$$\bar{r}_1 = \overset{\#}{R}(\ell_1)\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(\ell_1) + \overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{1\ell}(\ell_1) + \overset{\#}{C}_{12}\right]\overset{\#}{R}(-\ell_1)\,\bar{r}_2$$

$$\bar{r}_2 = \overset{\#}{R}(\ell_2)\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{1\ell}(\ell_2) + \overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{2\ell}(\ell_2) + \overset{\#}{C}_{12}\right]\overset{\#}{R}(-\ell_2)\,\bar{r}_3$$

$$\vdots \qquad\qquad \vdots$$

$$\bar{r}_{j-2}=\overset{\#}{R}(\ell_{j-2})\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-3)\ell}(\ell_{j-2})+\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})+\overset{\#}{C}_{12}\right]\overset{\#}{R}(-\ell_{j-2})\bar{r}_{j-1}$$

$$\bar{r}_{j-1}=\overset{\#}{R}(\ell_{j-1})\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})+\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})+\overset{\#}{C}_{12}\right]\overset{\#}{R}(-\ell_{j-1})\bar{r}_{j\ell}$$

$$\bar{r}_{j\ell}=\overset{\#}{R}(a_{jk})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(a_{jk})\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\right]^{-1}\left[\overset{\#}{\Gamma}_{jr}(a_{jk})\overset{\#}{C}_{+}+\overset{\#}{C}_{-}\right]\overset{\#}{C}_{f}^{-1}\overset{\approx}{P}(a_{jk})$$

$$\bar{q}_{jr}=\overset{\#}{R}(-a_{jk})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\overset{\#}{\Gamma}_{jr}(a_{jk})\right]^{-1}\left[\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\overset{\#}{C}_{-}+\overset{\#}{C}_{+}\right]\overset{\#}{C}_{f}^{-1}\overset{\approx}{P}(a_{jk})$$

$$\bar{q}_{j+1}=\overset{\#}{R}(-\ell_{j})\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_{j})\right]^{-1}\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{jr}(\ell_{j})\right]\overset{\#}{R}(\ell_{j})\bar{q}_{jr}$$

$$\bar{q}_{j+2}=\overset{\#}{R}(-\ell_{j+1})\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+2)r}(\ell_{j+1})\right]^{-1}\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_{j+1})\right]\overset{\#}{R}(\ell_{j+1})\bar{q}_{j+1}$$

$$\vdots \qquad\qquad \vdots$$

$$\bar{q}_{n-1}=\overset{\#}{R}(-\ell_{n-2})\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(n-1)r}(\ell_{n-2})\right]^{-1}\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(n-2)r}(\ell_{n-2})\right]\overset{\#}{R}(\ell_{n-2})\bar{q}_{n-2}$$

$$\bar{q}_{n}=\overset{\#}{R}(-\ell_{n-1})\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{n}(\ell_{n-1})\right]^{-1}\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(n-1)r}(\ell_{n-1})\right]\overset{\#}{R}(\ell_{n-1})\bar{q}_{n-1}\ . \tag{26a}$$

If a superposition technique is used, the total dynamic responses caused by all the driving forces distributed along the shaft can be obtained.

SOLUTION IN WAVE FORM

Proceeding with a lengthy matrix algebraic manipulation of the resulting Eqs. (26), as shown in the preceding section, and applying the superposition technique, one may express the total dynamic response caused by all driving forces $\overset{\approx}{P}(a_{jk})$, $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, k(j)$ along the shaft in the following traveling wave forms. (See the complete mathematical derivations in Appendix F.) Explanations of the traveling wave form solutions will be given in detail in Chapter 3.

In the i^{th} span, or on $\left[\ell_{i-1}, \ell_i\right]$, $i = 1, 2, \ldots, n$

$$\begin{bmatrix} \tilde{\tilde{Y}} \\ \tilde{\tilde{F}} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\bar{U}}_i(x) \\ \overset{\ggg}{\bar{U}}_i(x) \end{bmatrix} \tag{27}$$

where

$\overset{\ggg}{\bar{U}}(x)$ = total wave traveling to the right in the i^{th} span

$$= \sum_{j=1}^{n} \overset{\ggg}{\bar{U}}_{ij}(x)$$

$\overset{\lll}{\bar{U}}_i(x)$ = total wave traveling to the left in the i^{th} span

$$= \sum_{j=1}^{n} \overset{\lll}{\bar{U}}_{ij}(x) \quad .$$

If $j<i$, $j = 1, 2, \ldots, i-1$; then

$$\overset{\ggg}{\bar{U}}_{jj}(x) = \overset{\gg}{\bar{U}}_{jj}(x) + \overset{>}{\bar{U}}_{jj}(x)$$

$$\overset{\lll}{\bar{U}}_{jj}(x) = \overset{\ll}{\bar{U}}_{jj}(x) + \overset{<}{\bar{U}}_{jj}(x)$$

$$\overset{>}{\bar{U}}_{jj}(x) = \sum_{k=1}^{k(j)} \overset{\#}{R}(x-a_{jk})\overset{\#}{C}_+\overset{\#}{C}_f^{-1}\tilde{\tilde{P}}(a_{jk})\left[H(x-a_{jk})-H(x-\ell_j)\right]$$

$$\overset{<}{\bar{U}}_{jj}(x) = \sum_{k=1}^{k(j)} \overset{\#}{R}(a_{jk}-x)\overset{\#}{C}_-\overset{\#}{C}_f^{-1}\tilde{\tilde{P}}(a_{jk})\left[H(a_{jk}-x)-H(\ell_{j-1}-x)\right]$$

$$\overset{\gg}{\bar{U}}_{jj}(x)=\overset{\#}{R}(x-\ell_{j-1})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\right]^{-1}\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})$$

$$\times\left[\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{>}{\bar{U}}_{jj}(\ell_j)+\overset{<}{\bar{U}}_{jj}(\ell_{j-1})\right]\left[H(x-\ell_{j-1})-H(x-\ell_j)\right]$$

$$\overset{\ll}{\bar{U}}_{jj}(x)=\overset{\#}{R}(\ell_j-x)\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\right]^{-1}\overset{\#}{\Gamma}_{jr}(\ell_j)$$

$$\times\left[\overset{>}{\bar{U}}_{jj}(\ell_j)+\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{<}{\bar{U}}_{jj}(\ell_{j-1})\right]\left[H(x-\ell_{j-1})-H(x-\ell_j)\right]$$

$$\overset{\ggg}{\bar{U}}_{(j+1)j}(x)=\overset{\#}{R}(x-\ell_j)\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}\left[\overset{\gg}{\bar{U}}_{jj}(\ell_j)+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ll}{\bar{U}}_{jj}(\ell_j)\right]$$

$$\overset{\lll}{\bar{U}}_{(j+1)j}(x)=\overset{\#}{R}(\ell_{j+1}-x)\overset{\#}{\Gamma}_{(j+1)r}(\ell_{j+1})\overset{\#}{R}(\ell_{j+1}-\ell_j)\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}$$

$$\times\left[\overset{\gg}{\bar{U}}_{jj}(\ell_j)+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ll}{\bar{U}}_{jj}(\ell_j)\right]$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\overset{\ggg}{\bar{U}}_{ij}(x)=\overset{\#}{R}(x-\ell_{j-1})\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{ir}(\ell_{i-1})\right]^{-1}\left[\overset{\ggg}{\bar{U}}_{(i-1)j}(\ell_{i-1})+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\bar{U}}_{(i-1)j}(\ell_{i-1})\right]$$

$$\overset{\lll}{\bar{U}}_{ij}(x)=\overset{\#}{R}(\ell_i-x)\overset{\#}{\Gamma}_{ir}(\ell_i-\ell_{i-1})\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{ir}(\ell_{i-1})\right]^{-1}$$

$$\times\left[\overset{\ggg}{\bar{U}}_{(i-1)j}(\ell_{i-1})+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\bar{U}}_{(i-1)j}(\ell_{i-1})\right] .$$

If $j = i$, then

$$\overset{\ggg}{\bar{U}}_{ii}(x) = \overset{\gg}{\bar{U}}_{ii}(x) + \overset{>}{\bar{U}}_{ii}(x)$$

$$\overset{\lll}{\bar{U}}_{ii}(x) = \overset{\ll}{\bar{U}}_{ii}(x) + \overset{<}{\bar{U}}_{ii}(x)$$

$$\overset{>}{\bar{U}}_{ii}(x) = \sum_{k=1}^{k(i)} \overset{\#}{R}(x-a_{ik})\overset{\#}{C}_{+}\overset{\#}{C}_{f}^{-1}\overset{\simeq}{P}(a_{ik})\left[H(x-a_{ik})-H(x-\ell_i)\right]$$

$$\overset{<}{\bar{U}}_{ii}(x) = \sum_{k=1}^{k(i)} \overset{\#}{R}(a_{ik}-x)\overset{\#}{C}_{-}\overset{\#}{C}_{f}^{-1}\overset{\simeq}{P}(a_{ik})\left[H(a_{ik}-x)-H(\ell_{i-1}-x)\right]$$

$$\overset{\gg}{\bar{U}}_{ii}(x)=\overset{\#}{R}(x-\ell_{i-1})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(i-1)\ell}(\ell_{i-1})\overset{\#}{\Gamma}_{ir}(\ell_{i-1})\right]^{-1}\overset{\#}{\Gamma}_{(i-1)\ell}(\ell_{i-1})$$

$$\times\left[\overset{\#}{R}(\ell_i-\ell_{i-1})\overset{\#}{\Gamma}_{ir}(\ell_i)\overset{>}{\bar{U}}_{ii}(\ell_i)+\overset{<}{\bar{U}}_{ii}(\ell_{i-1})\right]\left[H(x-\ell_{i-1})-H(x-\ell_i)\right]$$

$$\overset{\ll}{\bar{U}}_{ii}(x)=\overset{\#}{R}(\ell_{i-x})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{ir}(\ell_i)\overset{\#}{\Gamma}_{(i-1)\ell}(\ell_i)\right]^{-1}\overset{\#}{\Gamma}_{ir}(\ell_i)$$

$$\times\left[\overset{>}{\bar{U}}_{ii}(\ell_i)+\overset{\#}{R}(\ell_i-\ell_{i-1})\overset{\#}{\Gamma}_{(i-1)\ell}(\ell_{i-1})\overset{<}{\bar{U}}_{ii}(\ell_{i-1})\right]\left[H(x-\ell_{i-1})-H(x-\ell_i)\right] .$$

If $j > i$, $j = n, n-1, \ldots, i+1$; then

$$\overset{\ggg}{\bar{U}}_{jj}(x) = \overset{\gg}{\bar{U}}_{jj}(x) + \overset{>}{\bar{U}}_{jj}(x)$$

$$\overset{\lll}{\bar{U}}_{jj}(x) = \overset{\ll}{\bar{U}}_{jj}(x) + \overset{<}{\bar{U}}_{jj}(x)$$

$$\overset{>}{\bar{U}}_{jj}(x) = \sum_{k=1}^{k(j)} \overset{\#}{R}(x-a_{jk})\overset{\#}{C}_{+}\overset{\#}{C}_{f}^{-1}\overset{\simeq}{P}(a_{jk})\left[H(x-a_{jk})-H(x-\ell_j)\right]$$

$$\overset{<}{\bar{U}}_{jj}(x) = \sum_{k=1}^{k(j)} \overset{\#}{R}(a_{jk}-x)\overset{\#}{C}_{-}\overset{\#}{C}_{f}^{-1}\overset{\simeq}{P}(a_{jk})\left[H(a_{jk}-x)-H(\ell_{j-1}-x)\right]$$

$$\overset{\gg}{\bar{U}}_{jj}(x)=\overset{\#}{R}(x-\ell_{j-1})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\right]^{-1}\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})$$

$$\times\left[\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{>}{\bar{U}}_{jj}(\ell_j)+\overset{<}{\bar{U}}_{jj}(\ell_{j-1})\right]\left[H(x-\ell_{j-1})-H(x-\ell_j)\right]$$

$$\overset{\ll}{\bar{U}}_{jj}(x)=\overset{\#}{R}(\ell_j-x)\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\right]^{-1}\overset{\#}{\Gamma}_{jr}(\ell_j)$$

$$\times\left[\overset{>}{\bar{U}}_{jj}(\ell_j)+\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{<}{\bar{U}}_{jj}(\ell_{j-1})\right]\left[H(x-\ell_{j-1})-H(x-\ell_j)\right]$$

$$\overset{\ggg}{\bar{U}}_{(j-1)j}=\overset{\#}{R}(x-\ell_{j-2})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})R(\ell_{j-1}-\ell_{j-2})\left[\overset{\#}{I}+\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})\right]^{-1}$$

$$\times\left[\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\bar{U}}_{jj}(\ell_{j-1})+\overset{\lll}{\bar{U}}_{jj}(\ell_{j-1})\right]$$

$$\overset{\lll}{\bar{U}}_{(j-1)j}=\overset{\#}{R}(\ell_{j-1}-x)\left[\overset{\#}{I}+\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})\right]^{-1}\left[\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\bar{U}}_{jj}(\ell_{j-1})+\overset{\lll}{\bar{U}}_{jj}(\ell_{j-1})\right]$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\overset{\ggg}{\bar{U}}_{ij}(x)=\overset{\#}{R}(x-\ell_{i-1})\overset{\#}{\Gamma}_{(i-1)\ell}(\ell_{i-1})\overset{\#}{R}(\ell_i-\ell_{i-1})\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(i-1)\ell}(\ell_i)\right]^{-1}$$

$$\times\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\bar{U}}_{(i+1)j}(\ell_i)+\overset{\lll}{\bar{U}}_{(i+1)j}(\ell_i)\right]$$

$$\overset{\lll}{\bar{U}}_{ij}(x)=\overset{\#}{R}(\ell_i-x)\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(i-1)\ell}(\ell_i)\right]^{-1}\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\bar{U}}_{(i+1)j}(\ell_i)+\overset{\lll}{\bar{U}}_{(i+1)j}(\ell_i)\right]$$

where

$$H(p) = \begin{cases} \overset{\#}{0} & \text{when } p < 0 \\ \overset{\#}{I} & \text{when } p > 0 \end{cases}$$

$\overset{>}{\bar{U}}$, $\overset{<}{\bar{U}}$ can be thought of as "incident waves" traveling to the right and left, respectively. They are independent of shaft support conditions.

$\overset{\gg}{\bar{U}}$, $\overset{\ll}{\bar{U}}$ can be thought of as "reflected waves" traveling to the right and left, respectively. They are dependent on shaft support conditions.

$\overset{\ggg}{\bar{U}}$ and $\overset{\lll}{\bar{U}}$ are the resultant of the incident and reflected waves traveling to the right and left, respectively. They may be thought of as "total waves".

CHAPTER 3

MANIPULATION OF THE SOLUTIONS

TRAVELING WAVE CONCEPT

The steady-state wave interpretation of the solution of Eqs. (27) provides a means toward visualizing the effects of support parameters (mass, spring, damping) and shaft characteristics on hypercritical shaft behavior. For the purpose of illustration, only one driving force, $\tilde{\tilde{P}}(a_{jk})$, will be considered in the following discussions.

By considering the j^{th} span first, i. e., on $\left[\ell_{j-1}, \ell_j\right]$, incident waves traveling along the shaft are considered as being initiated by the action of the single driving force, $\tilde{\tilde{P}}(a_{jk})$, located at $x = a_{jk}$. $\overset{>}{\overline{U}}_{jjk}(x)$ and $\overset{<}{\overline{U}}_{jjk}(x)$ are defined, as before, respectively, as incident waves traveling to the right and left from $x = a_{jk}$. The first subscript incidates the span in which the wave is traveling; the second subscript, the span from which the wave originates or the span in which the driving force is located; and the third subscript, the location of the driving force. For example, $U_{jjk}(x)$ means that this wave is traveling to the right on the j^{th} span; also, that the original wave originated in the j^{th} span, and that the driving force is located at $x = a_{jk}$. It can be seen from the following expressions that the incident waves are independent of support conditions. (See Eqs. (108) and (109) in Appendix F.)

$$\overset{>}{\overline{U}}_{jjk}(x)=\overset{\#}{R}(x-a_{jk})\overset{\#}{C}_{+}\overset{\#}{C}_f^{-1}\tilde{\tilde{P}}(a_{jk})\left[H(x-a_{jk})-H(x-\ell_j)\right] \tag{28}$$

exists only on $\left[a_{jk}, \ell_j\right]$

$$\overset{<}{\overline{U}}_{jjk}(x)=\overset{\#}{R}(a_{jk}-x)\overset{\#}{C}_{-}\overset{\#}{C}_f^{-1}\tilde{\tilde{P}}(a_{jk})\left[H(a_{jk}-x)-H(\ell_{j-1}-x)\right]$$

exists only on $\left[\ell_{j-1}, a_{jk}\right]$

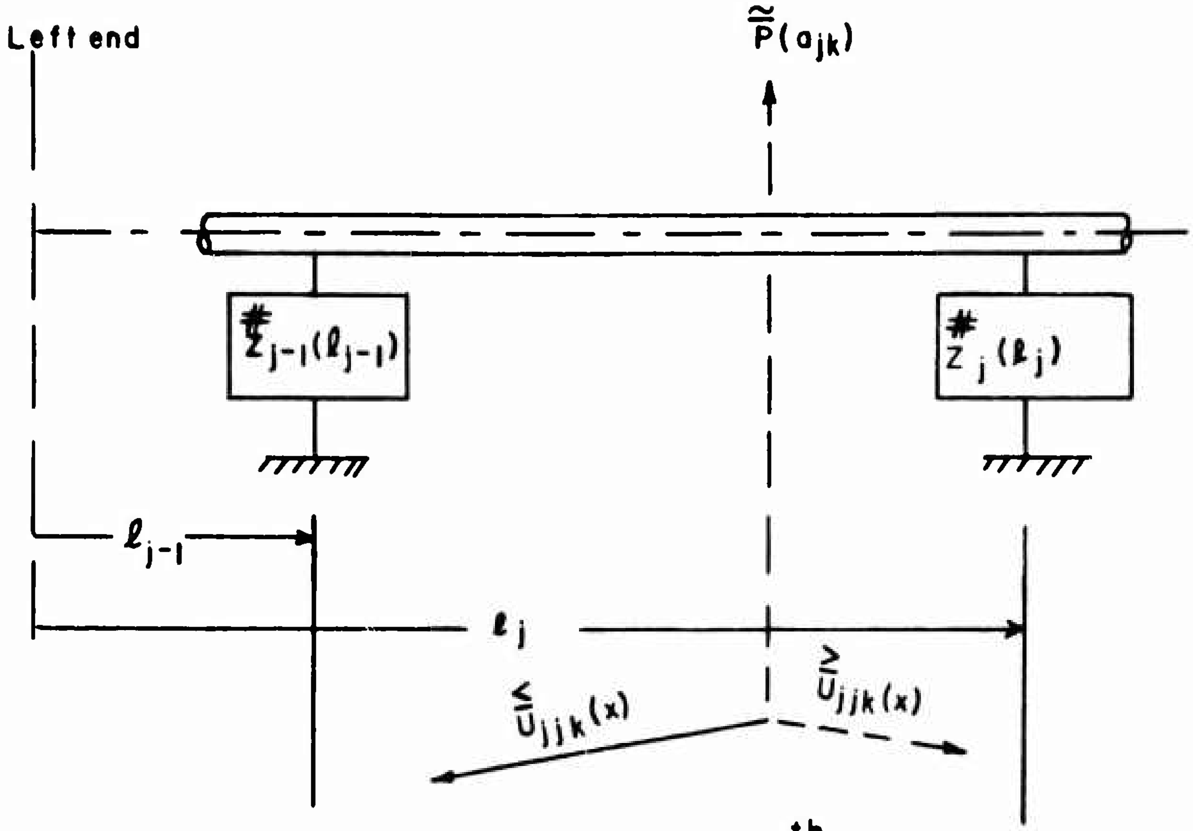

Figure 9. Incident Waves in the j^{th} Span.

With reference to Figure 9 and Eqs. (28) and (29), the propagation of the wave from the point of application, $x = a_{jk}$, of the driving force $\tilde{P}(a_{jk})$ is described by the "propagation matrix", $\overset{\#}{R}(x-a_{jk})$ or $\overset{\#}{R}(a_{jk}-x)$, as the wave is traveling to the right or left, respectively. For instance, at $x = \ell_j$, $\overset{>}{U}_{jjk}(x) = \overset{>}{U}_{jjk}(\ell_j)$ has traveled a distance $(\ell_j - a_{jk})$ to the j^{th} support of the system. The quantities $\left[H(x-a_{jk})-H(x-\ell_j)\right]$ and $\left[H(a_{jk}-x)-H(\ell_{j-1}-x)\right]$ simply indicate the regions in which the waves exist. The quantities $\overset{\#}{C}_+\overset{\#}{C}_f^{-1}$ and $\overset{\#}{C}_-\overset{\#}{C}_f^{-1}$ perform elementary transformations or operations on the applied driving force and are associated with frequency and shaft characteristics as defined in Appendix B.

A similar interpretation may be given to the quantities $\overset{\gg}{U}_{jjk}(x)$ and $\overset{\ll}{U}_{jjk}(x)$, which are regarded as reflected waves traveling to the right and left, respectively. They may be expressed as follows (see Eqs. (114) and (115) in Appendix F):

$$\overset{\gg}{\bar{U}}_{jjk}(x)=\overset{\#}{R}(x-\ell_{j-1})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\right]^{-1}\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})$$

$$\times\left[\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{>}{\bar{U}}_{jjk}(\ell_j)+\overset{<}{\bar{U}}_{jjk}(\ell_{j-1})\right]\left[H(x-\ell_{j-1})-H(x-\ell_j)\right] \qquad (30)$$

$$\overset{\ll}{\bar{U}}_{jjk}(x)=\overset{\#}{R}(\ell_j-x)\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\right]^{-1}\overset{\#}{\Gamma}_{jr}(\ell_j)$$

$$\times\left[\overset{>}{\bar{U}}_{jjk}(\ell_j)+\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{<}{\bar{U}}_{jjk}(\ell_{j-1})\right]\left[H(x-\ell_{j-1})-H(x-\ell_j)\right] \qquad (31)$$

Both exist only on $\left[\ell_{j-1},\ \ell_j\right]$.

The second term in Eq. (30) or Eq. (31) may be related to an infinite series:

$$\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\right]^{-1}$$

$$=\overbrace{\overset{\#}{I}}^{i}+\overbrace{\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})}^{ii}+\overbrace{\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})}^{iii}+$$

$$\overbrace{\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})}^{iv}+\ldots \qquad (32)$$

$$\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\right]^{-1}$$

$$=\overbrace{\overset{\#}{I}}^{i}+\overbrace{\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)}^{ii}+\overbrace{\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)}^{iii}+$$

$$\overbrace{\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)}^{iv}+\ldots \qquad (33)$$

If the first term is called i, the second ii, etc., as noted above, the expression for $\overset{\gg}{\bar{U}}_{jjk}(x)\Big|_{i}$ can be obtained. Again, $\left[H(x-\ell_{j-1})-H(x-\ell_j)\right]$ restricts the range of the waves to $\left[\ell_{j-1}, \ell_j\right]$.

Consider the first term i of the expansion for $\left[I-\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\right]^{-1}$,

$$\overset{\gg}{\bar{U}}_{jjk}(x)\Big|_{i}=\overset{\#}{R}(x-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{>}{\bar{U}}_{jjk}(\ell_j) +$$

$$\overset{\#}{R}(x-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{<}{\bar{U}}_{jjk}(\ell_{j-1}) \quad .$$

The last quantity, $\overset{<}{\bar{U}}_{jjk}(\ell_{j-1})$, represents an incident wave traveling to the left at the point $x = \ell_{j-1}$, i.e., the left end of the j^{th} span. If the quantity $\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})$ (which is a function of the configuration parameters of all the supports located to the left of $x = \ell_{j-1}+0$ and also of the associated shaft characteristics) signifies that this wave has been reflected at $x = \ell_{j-1}$, then the reflected wave now traveling to the right has propagated a distance $(x-\ell_{j-1})$ as indicated by $\overset{\#}{R}(x-\ell_{j-1})$. Simila- the last quantity of the first term, $\overset{>}{\bar{U}}_{jjk}(\ell_j)$, represents an incident which is traveling to the right and is located at $x = \ell_j$, i.e., the right support in the j^{th} span. The quantity $\overset{\#}{\Gamma}_{jr}(\ell_j)$, a function of the configuration parameters of all those supports located to the right of $x = \ell_j - 0$ and the associated shaft characteristics, signifies reflection of this wave to the left at $x = \ell_j$. The next term, $\overset{\#}{R}(\ell_j-\ell_{j-1})$, signifies propagation of the reflected wave to the left through a distance $(\ell_j-\ell_{j-1})$, i.e., length of the j^{th} span, until the wave reaches $x = \ell_{j-1}$. The wave is now reflected again to the right as indicated by $\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})$, and $\overset{\#}{R}(x-\ell_{j-1})$ signifies propagation through a distance $(x-\ell_{j-1})$ to the point under investigation. Thus, the term $\overset{\gg}{\bar{U}}_{jjk}(x)\Big|_{i}$ represents the contributions of two reflected waves to the right immediately following the initiation of the

incident waves, $\overset{>}{\bar{U}}_{jjk}(x)$ and $\overset{<}{\bar{U}}_{jjk}(x)$. This is illustrated in Figure 10.

Consider the second term, ii, of the expansion for

$$\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-i})\right]^{-1}$$

$$\left.\overset{\gg}{\bar{U}}_{jjk}(x)\right|_{ii}=\overset{\#}{R}(x-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(\ell_j-\ell_{j-1})$$
$$\times\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{>}{\bar{U}}_{jjk}(\ell_j)+$$
$$\overset{\#}{R}(x-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{<}{\bar{U}}_{jjk}(\ell_{j-1})$$

which may also be written as follows (see Eqs. (22)):

$$\left.\overset{\gg}{\bar{U}}_{jjk}(x)\right|_{ii}=\overset{\#}{R}(x-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-\ell_{j-1})$$
$$\times\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{>}{\bar{U}}_{jjk}(\ell_j)+$$
$$\overset{\#}{R}(x-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{<}{\bar{U}}_{jjk}(\ell_{j-1})\;.$$

If the same reasoning is used for the analysis of $\left.\overset{\gg}{\bar{U}}_{jjk}(x)\right|_{i}$, the added terms correspond to a wave propagating and reflecting an additional two more times, with the final reflected wave again traveling to the right. This is shown in Figure 11. Similar reasoning may be applied to the remaining terms. These remaining terms in the series, i.e., $\left.\overset{\gg}{\bar{U}}_{jjk}(x)\right|_{iii}$, $\left.\overset{\gg}{\bar{U}}_{jjk}(x)\right|_{iv}$, etc., together with $\left.\overset{\gg}{\bar{U}}_{jjk}(x)\right|_{i}$ and $\left.\overset{\gg}{\bar{U}}_{jjk}(x)\right|_{ii}$ account for all of the reflected waves traveling to the right. The summation of all of the waves traveling to the right is complete if the incident wave is added to the above; i.e.,

$$\begin{aligned}\overset{\ggg}{\bar{U}}_{jjk}(x)&=\overset{>}{\bar{U}}_{jjk}(x)+\left.\overset{\gg}{\bar{U}}_{jjk}(x)\right|_{i}+\left.\overset{\gg}{\bar{U}}_{jjk}(x)\right|_{ii}+\left.\overset{\gg}{\bar{U}}_{jjk}(x)\right|_{iii}+\ldots\\&=\overset{>}{\bar{U}}_{jjk}(x)+\overset{\gg}{\bar{U}}_{jjk}(x)\;.\end{aligned}\tag{34}$$

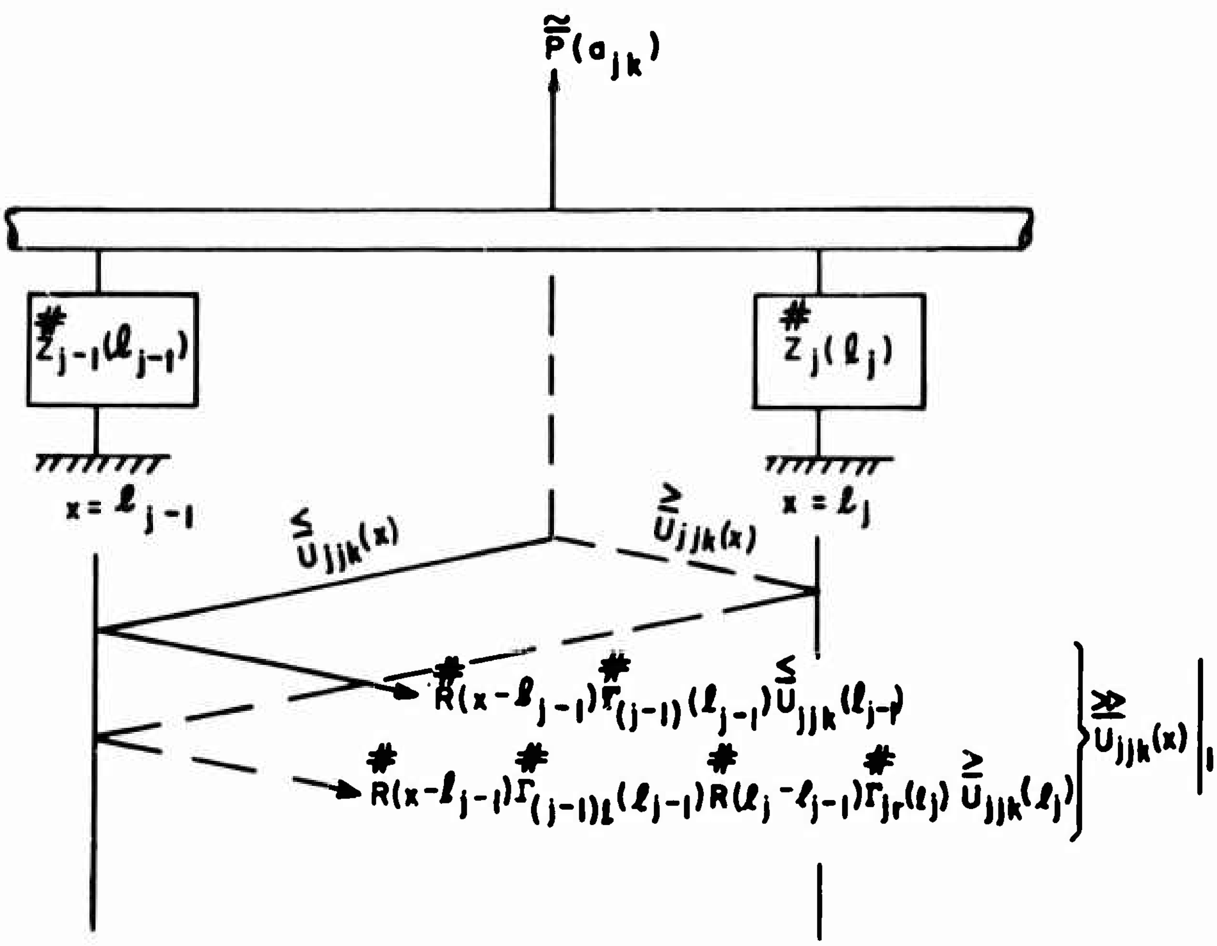

Figure 10. First-term Propagation of Reflected Waves in the $[\ell_{j-1}, \ell_j]$ Portion of the Shaft.

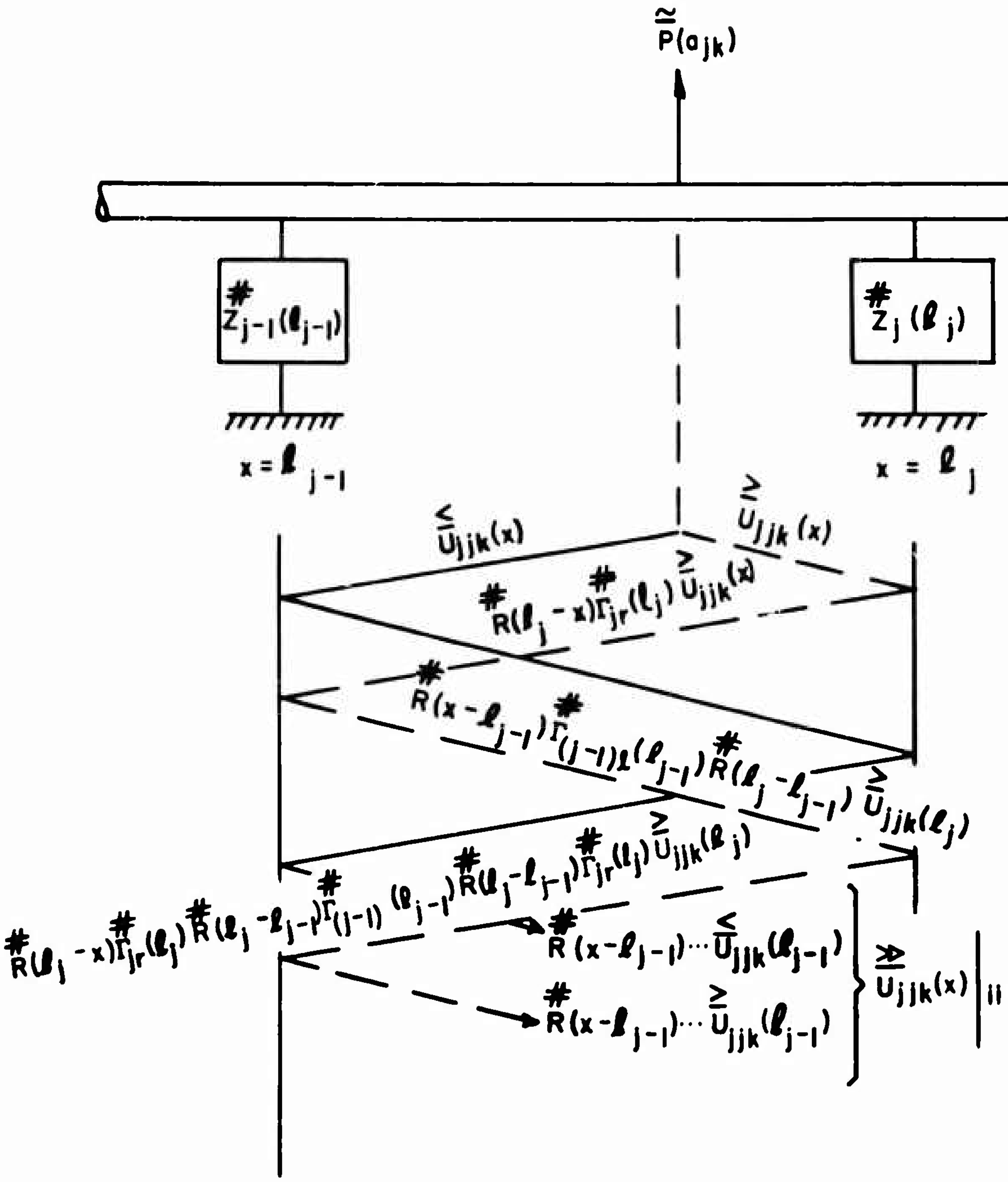

Figure 11. First- and Second-term Propagation of Waves in the $[\ell_{j-1}, \ell_j]$ Portion of the Shaft.

Similar results may be obtained for $\overset{\ll}{\bar{U}}_{jjk}(x)$. That is,

$$\overset{\ll}{\bar{U}}_{jjk}(x)=\overset{\ll}{\bar{U}}_{jjk}(x)\Big|_{i}+\overset{\ll}{\bar{U}}_{jjk}(x)\Big|_{ii}+\overset{\ll}{\bar{U}}_{jjk}(x)\Big|_{iii}+\ \cdot\ \cdot\ \cdot \tag{35}$$

The summation of $\overset{\ll}{\bar{U}}_{jjk}(x)$, the reflected wave traveling to the left in the j^{th} span, with $\overset{<}{\bar{U}}_{jjk}(x)$, the incident wave traveling to the left, accounts for all the waves traveling to the left as they are observed passing point x in the j^{th} span of the shaft.

The complete response on $\left[\ell_{j-1},\ \ell_j\right]$ at point x can be determined by properly combining all of the waves traveling past point x in both the left and right directions (see Eq. (127) in Appendix F). That is,

$$\left.\begin{aligned}
&\text{On } \left[\ell_{j-1},\ \ell_j\right],\\
&\begin{bmatrix}\tilde{Y}\\ \tilde{F}\end{bmatrix}=\begin{bmatrix}\frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0}\\ \overset{\#}{0} & \overset{\#}{C}_f\end{bmatrix}\begin{bmatrix}\overset{\#}{C}_{11} & \overset{\#}{C}_{12}\\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22}\end{bmatrix}\begin{bmatrix}\overset{\ggg}{\bar{U}}_{jjk}(x)\\ \overset{\lll}{\bar{U}}_{jjk}(x)\end{bmatrix}\\
&\text{or}\\
&\bar{Y}(x)=\frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\ggg}{\bar{U}}_{jjk}(x)+\overset{\#}{C}_{12}\overset{\lll}{\bar{U}}_{jjk}(x)\right]\\
&\bar{F}(x)=\overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\ggg}{\bar{U}}_{jjk}(x)+\overset{\#}{C}_{22}\overset{\lll}{\bar{U}}_{jjk}(x)\right]\ .
\end{aligned}\right\} \tag{36}$$

Eq. (36) indicates a relatively direct manner in which the waves are combined to obtain the total response at point x.

By examining the portion of the shaft on $\left[\ell_j,\ \ell_{j+1}\right]$ in terms of a single driving force $\tilde{P}(a_{jk})$, $\overset{\ggg}{\bar{U}}_{(j+1)jk}(x)$ and $\overset{\lll}{\bar{U}}_{(j+1)jk}(x)$ may be thought of as "modified total waves" which are initiated by the driving force at $x = a_{jk}$. As soon as the total waves, $\overset{\ggg}{\bar{U}}_{jjk}(x)$ and $\overset{\lll}{\bar{U}}_{jjk}(x)$, on $\left[\ell_{j-1},\ \ell_j\right]$ travel to $x = \ell_j$, the waves are modified or refracted by the j^{th} support and pass through as indicated by $\overset{\ggg}{\bar{U}}_{(j+1)jk}(x)$ and $\overset{\lll}{\bar{U}}_{(j+1)jk}(x)$ traveling in the $(j+1)^{th}$ span (see Eqs. (138) and (139) in Appendix F). Thus,

$$\overset{\ggg}{\bar{U}}_{(j+1)jk}(x)=\overset{\#}{R}(x-\ell_j)\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}\left[\overset{\ggg}{\bar{U}}_{jjk}(\ell_j)+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\bar{U}}_{jjk}(\ell_j)\right] \tag{37}$$

$$\overset{\lll}{\bar{U}}_{(j+1)jk}(x)=\overset{\#}{R}(\ell_{j+1}-x)\overset{\#}{\Gamma}_{(j+1)r}(\ell_{j+1})\overset{\#}{R}(\ell_{j+1}-\ell_j)\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}$$

$$\times\left[\overset{\ggg}{\bar{U}}_{jjk}(\ell_j)+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\bar{U}}_{jjk}(\ell_j)\right] \tag{38}$$

where, as before, the first subscript, $j+1$, indicates that the wave is traveling in the $(j+1)^{th}$ span; the second subscript, j, indicates that the original wave originates in the j^{th} span where the driving force is located; the third subscript, k, indicates the exact location of the driving force.

An explanation of traveling wave behavior similar to the previous discussion may be applied to $\overset{\ggg}{\bar{U}}_{(j+1)jk}(x)$ and $\overset{\lll}{\bar{U}}_{(j+1)jk}(x)$ in the $(j+1)^{th}$ span. Note that the second term of $\overset{\ggg}{\bar{U}}_{(j+1)jk}(x)$ can be restated as an infinite series:

$$\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}$$

$$=\overset{i}{\overset{\#}{I}}+\overbrace{(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)}^{ii}+\overbrace{(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)}^{iii}+$$

$$\overbrace{(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)}^{iv}+\ldots \tag{39}$$

Again, the order of the terms is indicated by i, ii, iii, etc. Considering $\overset{\ggg}{\bar{U}}_{(j+1)jk}(x)$ and taking, for the time being, only the first term i of the above expression,

$$\overset{\ggg}{\bar{U}}_{(j+1)jk}(x)\Big|_i=\overset{\#}{R}(x-\ell_j)\left[\overset{\ggg}{\bar{U}}_{jjk}(\ell_j)+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\bar{U}}_{jjk}(\ell_j)\right],$$

where the last term signifies that the total wave in the j^{th} span passes through the j^{th} support and enters into the $(j+1)^{th}$ span as a modified wave. $\overset{\#}{R}(x-\ell_j)$ signifies that the modified wave has propagated to the right a distance $(x-\ell_j)$. This is illustrated in Figure 12.

Consider now the second term, ii, of the expression for $\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}$,

$$\overset{>>>}{\overline{U}}_{(j+1)jk}(x)\Big|_{ii}=\overset{\#}{R}(x-\ell_j)(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\left[\overset{>>>}{\overline{U}}_{jjk}(\ell_j)+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{<<<}{\overline{U}}_{jjk}(\ell_j)\right]$$

which may be rewritten as follows (see Eqs. (22)):

$$\overset{>>>}{\overline{U}}_{(j+1)jk}(x)\Big|_{ii}=\overset{\#}{R}(x-\ell_j)(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{R}(\ell_{j+1}-\ell_j)\overset{\#}{\Gamma}_{(j+1)r}(\ell_{j+1})\overset{\#}{R}(\ell_{j+1}-\ell_j)$$
$$\times\left[\overset{>>>}{\overline{U}}_{jjk}(\ell_j)+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{<<<}{\overline{U}}_{jjk}(\ell_j)\right] .$$

If the same reasoning as before is followed, the added terms correspond to the wave propagating to the right after being reflected twice, first at the $(j+1)^{th}$ support and then at the j^{th} support. Appendix G shows that the term $(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})$ corresponds to a reflection matrix of a fixed end support. This means that the j^{th} support acts as a one-way fixed support which completely reflects modified waves coming from the right after being reflected back from the $(j+1)^{th}$ support, but which permits waves traveling from the left to pass through. This is illustrated in Figure 13. The same analysis may be applied to the remaining terms of the series. Also, similar reasoning may be applied to $\overset{<<<}{\overline{U}}_{(j+1)jk}(x)$, the modified wave traveling to the left as indicated in Eq. (38).

The complete response at point x on $\left[\ell_j, \ell_{j+1}\right]$ due to $\overset{\simeq}{P}(a_{jk})$ can be determined by properly combining all of the waves traveling past point x in both the left and right directions (see Eq. (140) in Appendix F). That is,

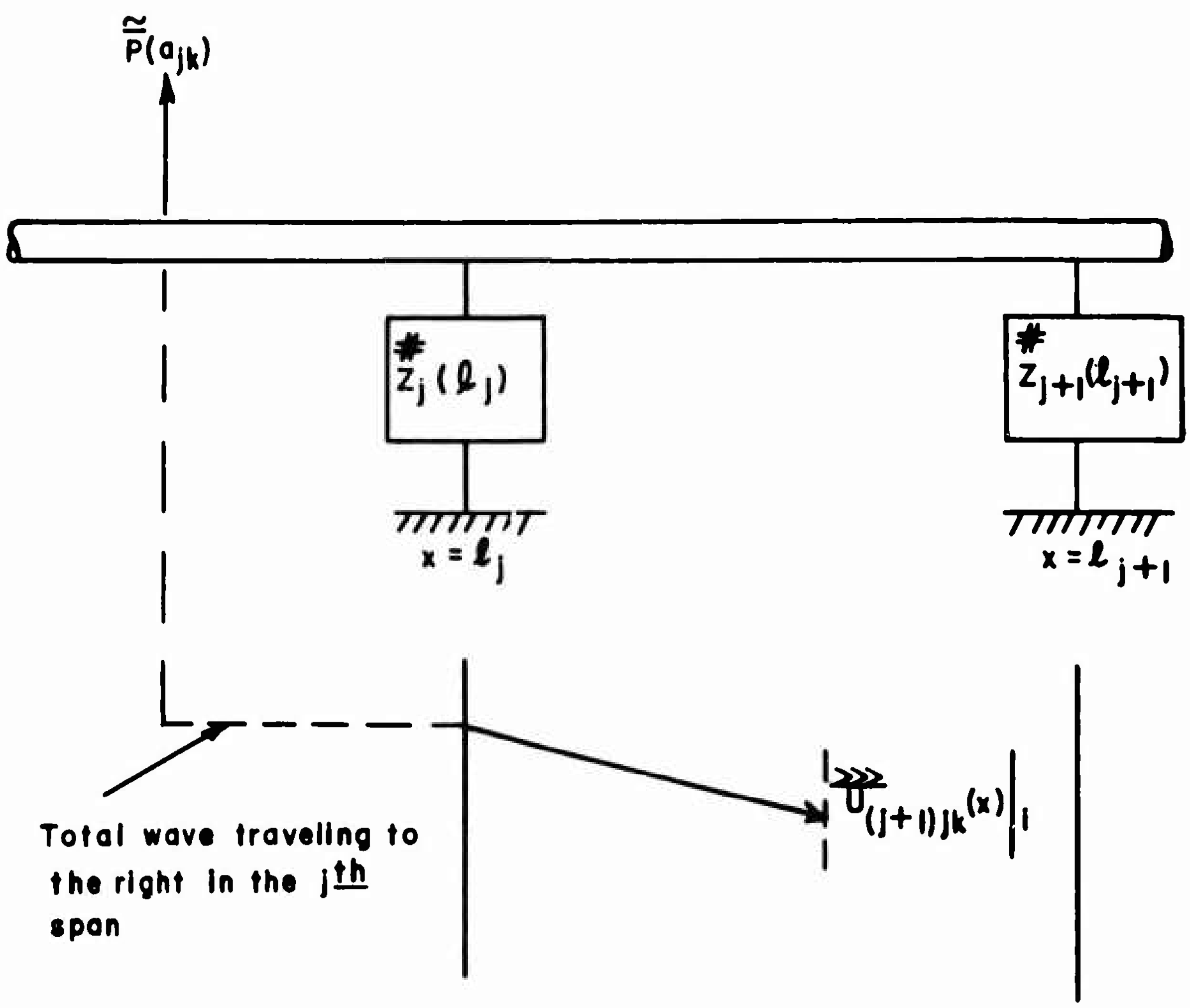

Figure 12. First-term Propagation of Modified Total Wave Traveling to the Right in the $[\ell_j, \ell_{j+1}]$ Portion of the Shaft.

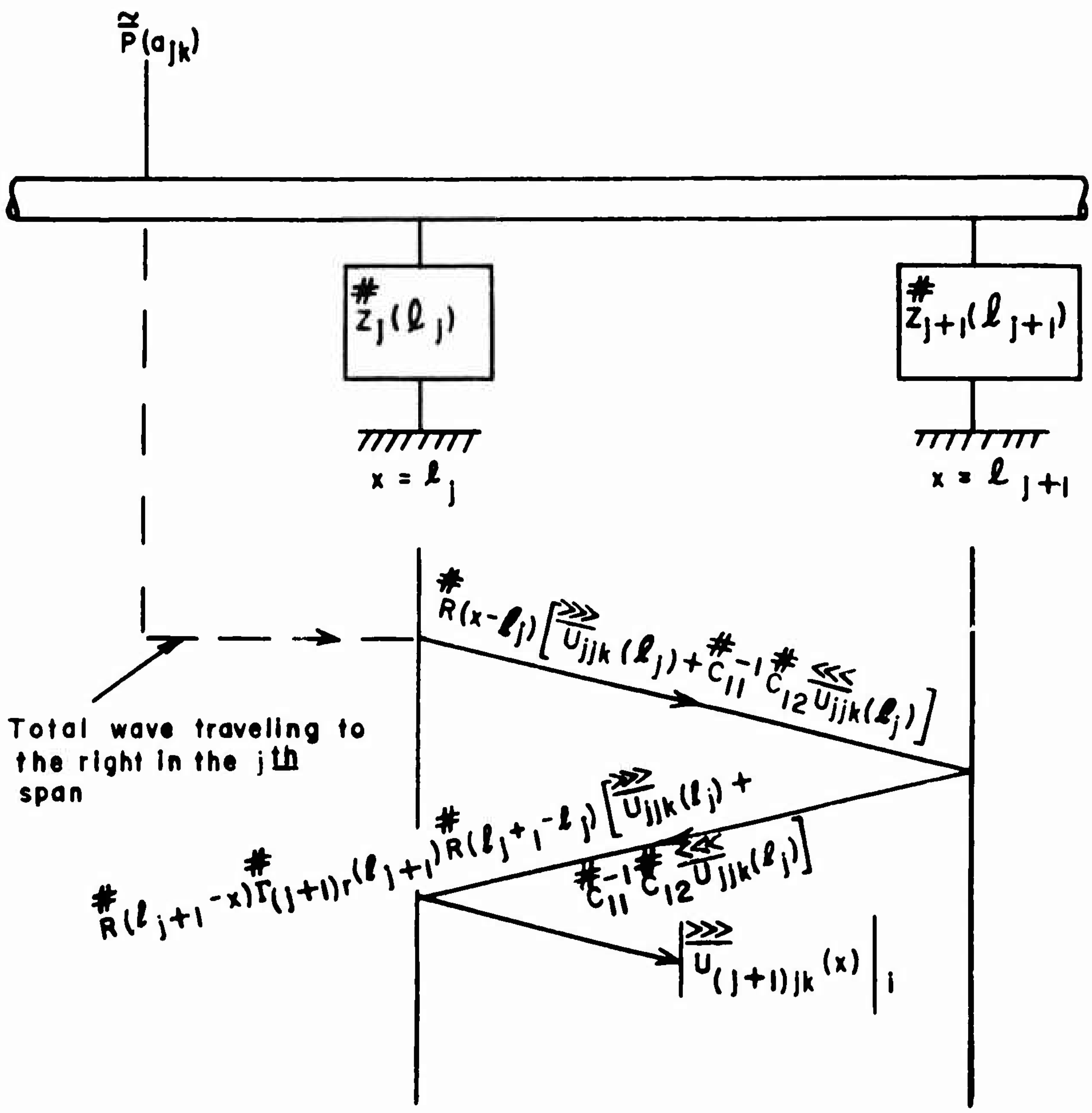

Figure 13. Second-term Propagation of Modified Total Wave Traveling to the Right in the $\left[\ell_j, \ell_{j+1}\right]$ Portion of the Shaft.

on $\left[\ell_j,\ \ell_{j+1}\right]$,

$$\left.\begin{aligned}
&\begin{bmatrix}\overset{\simeq}{Y}\\ \overset{\simeq}{F}\end{bmatrix}=\begin{bmatrix}\frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0}\\ \overset{\#}{0} & \overset{\#}{C}_f\end{bmatrix}\begin{bmatrix}\overset{\#}{C}_{11} & \overset{\#}{C}_{12}\\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22}\end{bmatrix}\begin{bmatrix}\overset{\ggg}{\bar{U}}_{(j+1)jk}(x)\\ \overset{\lll}{\bar{U}}_{(j+1)jk}(x)\end{bmatrix}\\
&\text{or}\\
&\overset{\simeq}{Y}(x)=\frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\ggg}{\bar{U}}_{(j+1)jk}(x)+\overset{\#}{C}_{12}\overset{\lll}{\bar{U}}_{(j+1)jk}(x)\right]\\
&\overset{\simeq}{F}(x)=\overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\ggg}{\bar{U}}_{(j+1)jk}(x)+\overset{\#}{C}_{22}\overset{\lll}{\bar{U}}_{(j+1)jk}(x)\right]\ .
\end{aligned}\right\}\quad(40)$$

The same manipulation of traveling wave concept used for the explanation of the dynamic response in the $(j+1)^{th}$ span may be extended to explain the dynamic response in the $(j+2)^{th}$ span as well; it also can be extended to explain the response in the $(j+3)^{th}$ span, and so forth, until the n^{th} span is reached. Similar argument also can be used for the dynamic response in the $(j-1)^{th}$ span, in which the modified waves (see Eqs. (135) and (136) in Appendix F) are as follows:

$$\begin{aligned}
\overset{\ggg}{\bar{U}}_{(j-1)jk}(x)=&\overset{\#}{R}(x-\ell_{j-2})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})\overset{\#}{R}(\ell_{j-1}-\ell_{j-2})\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})\right]^{-1}\\
&\times\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\bar{U}}_{jjk}(\ell_{j-1})+\overset{\lll}{\bar{U}}_{jjk}(\ell_{j-1})\right]
\end{aligned}\qquad(41)$$

$$\begin{aligned}
\overset{\lll}{\bar{U}}_{(j-1)jk}(x)=&\overset{\#}{R}(\ell_{j-1}-x)\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})\right]^{-1}\\
&\times\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\bar{U}}_{jjk}(\ell_{j-1})+\overset{\lll}{\bar{U}}_{jjk}(\ell_{j-1})\right]\ .
\end{aligned}\qquad(42)$$

The term $\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})\right]^{-1}$ can be restated in terms of an infinite series as

$$\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})\right]^{-1}$$

$$=\overset{\#}{I}+(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})+(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})+$$

$$(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})+\dots \tag{43}$$

Again, the order of the terms is indicated by i, ii, iii, etc. Considering $\overset{\gg}{\overline{U}}_{(j+1)jk}(x)$ and taking, for the time being, only the first term i of the above expression,

$$\overset{\gg}{\overline{U}}_{(j-1)jk}(x)\Big|_{i}=\overset{\#}{R}(x-\ell_{j-2})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})\overset{\#}{R}(\ell_{j-1}-\ell_{j-2})$$

$$\times\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\gg}{\overline{U}}_{jjk}(\ell_{j-1})+\overset{\ll}{\overline{U}}_{jjk}(\ell_{j-1})\right]$$

where the last term signifies that the total wave in the j^{th} span passes through the $(j-1)^{th}$ support and enters into the $(j-2)^{th}$ span with modified wave. $\overset{\#}{R}(\ell_{j-1}-\ell_{j-2})$ signifies that the modified wave has propagated a distance $(\ell_{j-1}-\ell_{j-2})$ and then is reflected to the right by the $(j-2)^{th}$ support, as indicated by the reflection matrix $\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})$. $\overset{\#}{R}(x-\ell_{j-2})$ again signifies that the reflected modified wave has propagated a distance $(x-\ell_{j-2})$, and then the wave is at the point under investigation. This is illustrated in Figure 14.

Consider now the second term, ii, of Eq. (41), corresponding to the ii term of the expansion of

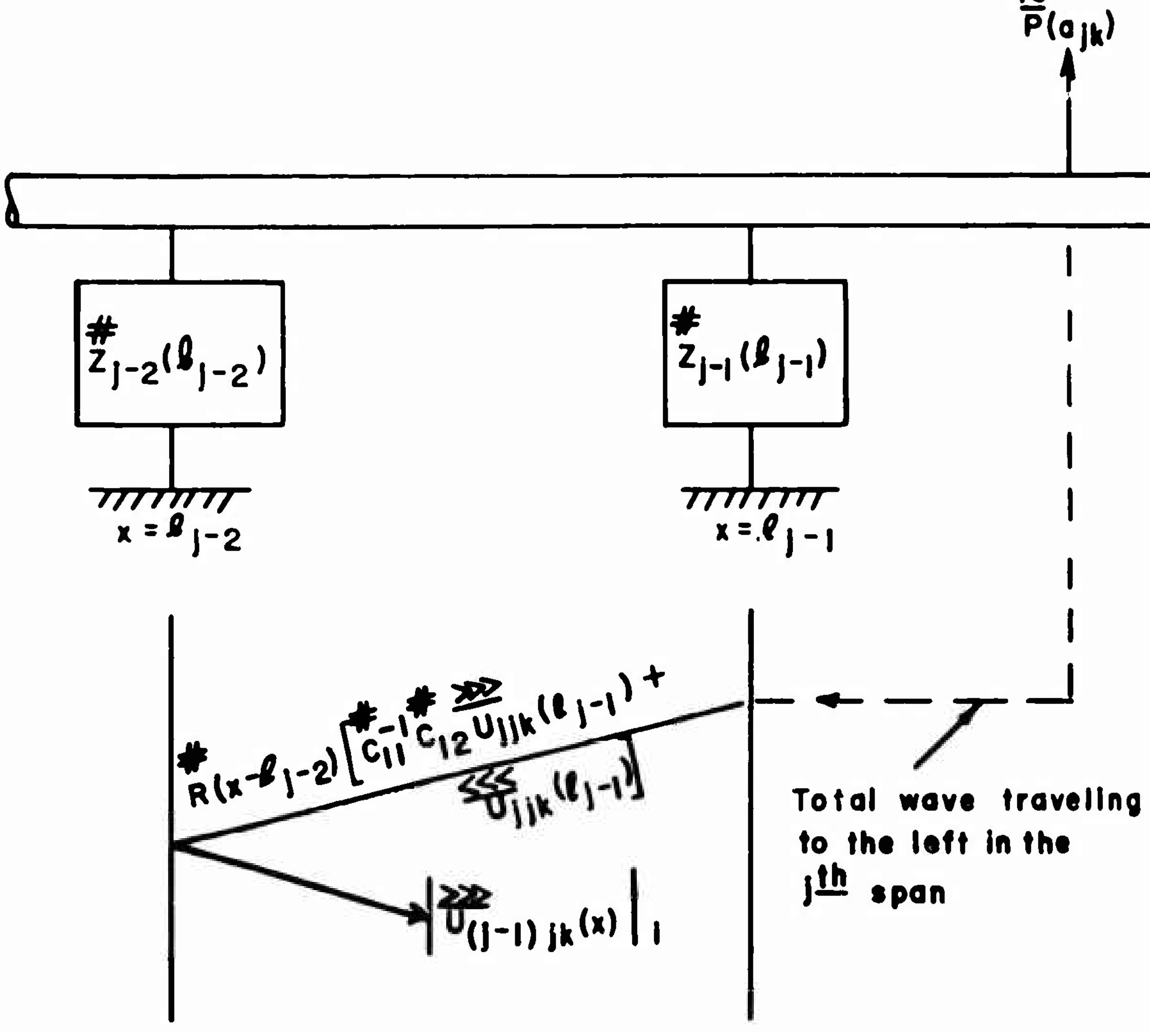

Figure 14. First-term Propagation of Modified Total Wave Traveling to the Right in the $[\ell_{j-2}, \ell_{j-1}]$ Portion of the Shaft.

$$\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})\right]^{-1} ,$$

$$\overset{\ggg}{\overline{U}}_{(j-1)jk}(x)\Big|_{ii}=\overset{\#}{R}(x-\ell_{j-2})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})\overset{\#}{R}(\ell_{j-1}-\ell_{j-2})(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})$$

$$\times\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\overline{U}}_{jjk}(\ell_{j-1})+\overset{\lll}{\overline{U}}_{jjk}(\ell_{j-1})\right]$$

which may be rewritten (see Eq. (22)) as follows:

$$\overset{\ggg}{\overline{U}}_{(j-1)jk}(x)\Big|_{ii}=\overset{\#}{R}(x-\ell_{j-2})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})\overset{\#}{R}(\ell_{j-1}-\ell_{j-2})(-\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12})\overset{\#}{R}(\ell_{j-1}-\ell_{j-2})$$

$$\times\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})\overset{\#}{R}(\ell_{j-1}-\ell_{j-2})\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\overline{U}}_{jjk}(\ell_{j-1})+\overset{\lll}{\overline{U}}_{jjk}(\ell_{j-1})\right] .$$

If the same reasoning as before is used, the added terms correspond to the wave propagating to the right after being reflected two more times. Again, the $(j-1)^{th}$ support acts as a one-way fixed support. This is illustrated in Figure 15. The same analysis may be applied to the remaining terms of the series. Similar reasoning may be applied also to $\overset{\lll}{\overline{U}}_{(j-1)jk}(x)$, the total wave traveling to the right.

The complete response at point x on $\left[\ell_{j-2}, \ell_{j-1}\right]$ due to $\overset{\simeq}{P}(a_{jk})$ can be determined by properly combining all of the waves traveling past point x in both the left and right directions (see Eq. (137) in Appendix F). That is,

$$\left.\begin{aligned}
&\text{on } \left[\ell_{j-2}, \ell_{j-1}\right] , \\
&\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\overline{U}}_{(j-1)jk}(x) \\ \overset{\lll}{\overline{U}}_{(j-1)jk}(x) \end{bmatrix} \\
&\overset{\simeq}{Y}(x)=\frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\ggg}{\overline{U}}_{(j-1)jk}(x)+\overset{\#}{C}_{12}\overset{\lll}{\overline{U}}_{(j-1)jk}(x)\right] \\
&\overset{\simeq}{F}(x)=\overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\ggg}{\overline{U}}_{(j-1)jk}(x)+\overset{\#}{C}_{22}\overset{\lll}{\overline{U}}_{(j-1)jk}(x)\right] .
\end{aligned}\right\} \quad (44)$$

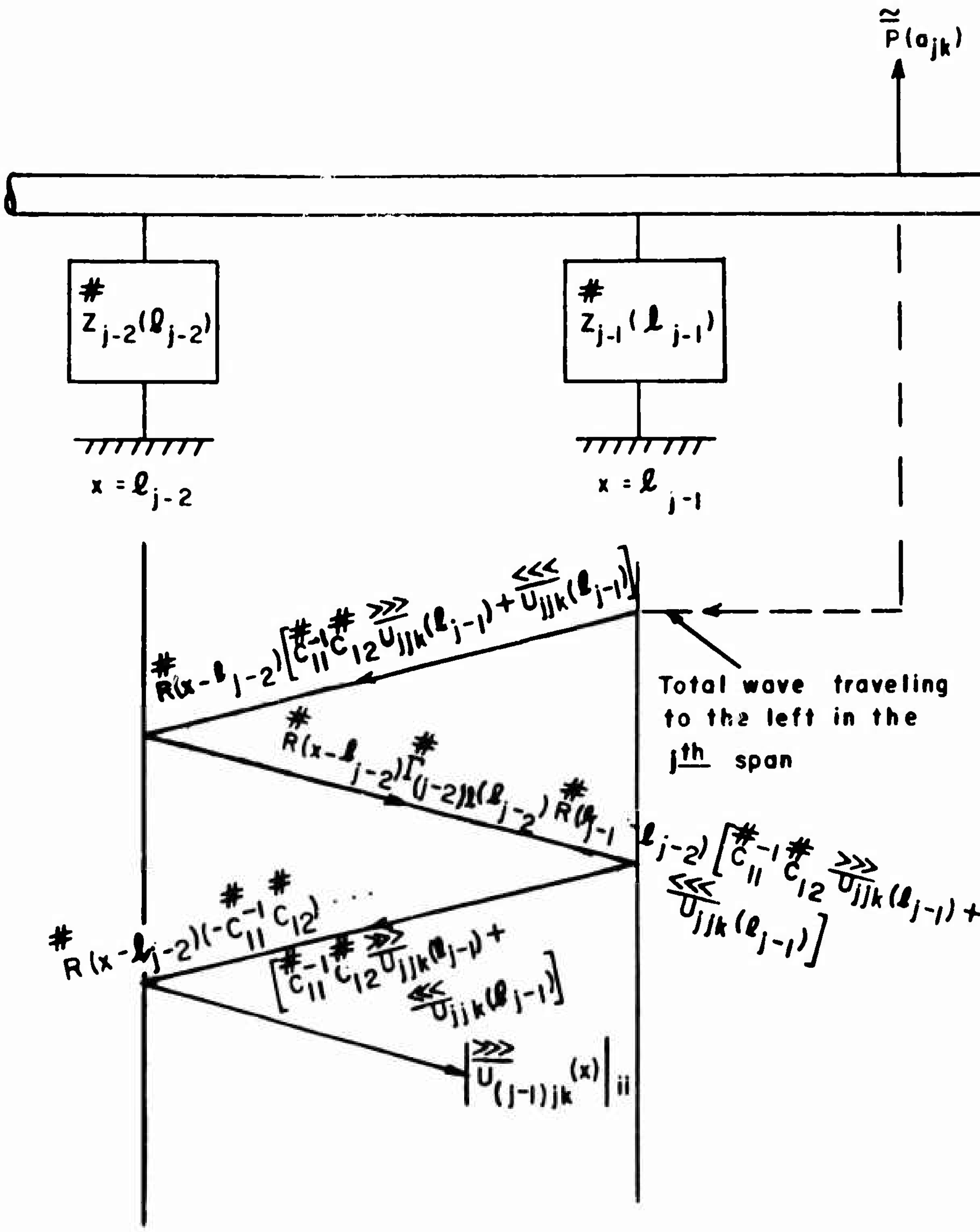

Figure 15. Second-term Propagation of Modified Total Wave Traveling to the Right in the $[\ell_{j-2}, \ell_{j-1}]$ Portion of the Shaft.

The same manipulation of traveling wave concept used for the explanation of the dynamic response in the $(j-1)^{th}$ span may be extended to explain the dynamic response in the $(j-2)^{th}$ span as well; it also can be extended to explain the response in the $(j-3)^{th}$ span, and so forth, until the 1^{st} span is reached.

The above analysis is based on the assumption that a single driving force, $\overset{\simeq}{P}(a_{jk})$, acts on the shaft. For driving forces distributed along the shaft, the traveling wave solution for each individual driving force is combined to give the total response at any point x by direct superposition. Complete solutions are given in Chapter 2 under "Solution in Wave Form".

IMPEDANCE MATCHING

The dynamic response of the shaft system shown in Figure 4 in terms of distributed driving forces, $\overset{\simeq}{P}(a_{jk})$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, k(j)$ is expressed by Eq. (27) in Chapter 2. In the i^{th} span, or on $[\ell_{i-1}, \ell_i]$, $i = 1, 2, \ldots, n$

$$\begin{bmatrix}\overset{\simeq}{Y}\\ \overset{\simeq}{F}\end{bmatrix} = \begin{bmatrix}\frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0}\\ \overset{\#}{0} & \overset{\#}{C}_f\end{bmatrix}\begin{bmatrix}\overset{\#}{C}_{11} & \overset{\#}{C}_{12}\\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22}\end{bmatrix}\begin{bmatrix}\overset{\ggg}{\bar{U}}_i(x)\\ \overset{\lll}{\bar{U}}_i(x)\end{bmatrix}.$$

By proper control of the conditions at both ends and intermediate supports, the reflected waves and modified waves may be eliminated, or at least minimized, such that the dynamic response consists solely, or principally, of incident waves which are independent of support configurations. The mathematical representation of the above statement is as follows:

$$\overset{\ggg}{\bar{U}}_i(x) \equiv \overset{>}{\bar{U}}_i(x) \quad \text{and} \quad \overset{\lll}{\bar{U}}_i(x) \equiv \overset{<}{\bar{U}}_i(x)$$

or

$$\overset{\gg}{\bar{U}}_i(x) = \overset{\ll}{\bar{U}}_i(x) \equiv 0 \tag{45}$$

where $i = 1, 2, \ldots, n$.

It can be seen from examination of the reflected and modified wave terms

in Eq. (27) that the condition $\overset{\gg}{\bar{U}}_i(x) = \overset{\ll}{\bar{U}}_i(x) \equiv 0$, $i = 1, 2, \ldots, n$, is satisfied by setting

$$\overset{\#}{\Gamma}_0(0)=\overset{\#}{\Gamma}_{1\ell}(\ell_1)=\overset{\#}{\Gamma}_{2\ell}(\ell_2)= \ldots =\overset{\#}{\Gamma}_{(n-1)\ell}(\ell_{n-1})\equiv \overset{\#}{0}$$

and

$$\overset{\#}{\Gamma}_n(\ell_n)=\overset{\#}{\Gamma}_{(n-1)r}(\ell_{n-1})=\overset{\#}{\Gamma}_{(n-2)r}(\ell_{n-2})= \ldots =\overset{\#}{\Gamma}_{1r}(\ell_1)\equiv \overset{\#}{0} \quad . \tag{46}$$

Physically, this is equivalent to setting the impedance at each support looking both to the left and to the right of the support point equal to the shaft's characteristic impedance. This manipulation is analogous to load matching in electric transmission line theory, in which a line is terminated by a load equal to its characteristic impedance; i.e., there is no reflected voltage wave. The proper ratio of input to output voltage in the incident wave is satisfied at the boundary, such that there is no need for the presence of a reflected wave to satisfy the boundary conditions. The existence of reflection waves stems from a need for satisfying boundary conditions. A mathematical verification of the above statement may be obtained from Eqs. (23) and (24), which are expressions of impedance in terms of reflection matrices and shaft characteristic impedance as shown in Chapter 2 on pages 26 and 27 . These relationships show that by letting all reflection matrices vanish, i.e., conditions of Eqs. (46), the following set of equations holds:

$$\overset{\Delta}{z}_0(0)=\overset{\Delta}{z}_{1\ell}(\ell_1)=\overset{\Delta}{z}_{2\ell}(\ell_2)= \ldots =\overset{\Delta}{z}_{(n-1)\ell}(\ell_{n-1})=\overset{\#}{C}_{21}\overset{\#}{C}{}_{11}^{-1}$$

and

$$\overset{\#}{z}_n(\ell_n)=\overset{\#}{z}_{(n-1)r}(\ell_{n-1})=\overset{\#}{z}_{(n-2)r}(\ell_{n-2})= \ldots =\overset{\#}{z}_{1r}(\ell_1)=\overset{\#}{C}_{21}\overset{\#}{C}{}_{11}^{-1} \tag{47}$$

where $\overset{\#}{z}_s = \overset{\#}{C}_{21}\overset{\#}{C}{}_{11}^{-1}$. The quantity $\overset{\#}{z}_s$ is called the characteristic impedance of the rotating shaft. In this case, all impedances appearing in Eq. (47) are termed "matching" or matched impedances.

If Eqs. (46) or Eqs. (47) are satisfied by proper control of all support conditions, the solutions of the dynamic response of hypercritical shafts in wave form due to distributed driving forces as expressed in Eq. (27) may be simplified as follows:

In the i^{th} span, or on $\left[\ell_{i-1}, \ell_i\right]$, $i = 1, 2, \ldots, n$

$$
\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{>}{\overline{U}}_i(x) \\ \overset{<}{\overline{U}}_i(x) \end{bmatrix} \tag{48}
$$

where

$$
\overset{>}{\overline{U}}_i(x) = \sum_{j=1}^{n} \overset{>}{\overline{U}}_{ij}(x)
$$

$$
\overset{<}{\overline{U}}_i(x) = \sum_{j=1}^{n} \overset{<}{\overline{U}}_{ij}(x) \quad .
$$

Note that the expression for $\overset{>}{\overline{U}}_{ij}(x)$ and $\overset{<}{\overline{U}}_{ij}(x)$ are given in Eq. (27), except that each wave function $\overset{<}{\overline{U}}_{(j+1)j}(x)$, . . . , $\overset{<}{\overline{U}}_{ij}(x)$, and $\overset{>}{\overline{U}}_{(j-1)j}(x)$, . . . , $\overset{>}{\overline{U}}_{ij}(x)=\overset{\#}{0}$.

DISCUSSIONS ON IMPEDANCE MATCHING

Examination of Eq. (48) in the preceding section permits the following observations to be made. If the impedances looking both to the left and to the right at each support are matched to the shaft's characteristic impedance or, in other words, if all the reflection matrices looking both to the left and to the right at any support are equal to null matrices, the dynamic response will contain only the incident waves due to driving forces. This means that incident waves simply pass through all supports without any modification. This is consistent with the mathematical explanation, since the incident waves are independent of support conditions. Hence, Eqs. (46) or Eqs. (47) may serve as a criterion for the determination of the support conditions for shafts which are subjected to incident waves only. Supports which satisfy Eqs. (46) or Eqs. (47) are termed optimized, since the dynamic response is minimized; no amplitude buildup due to reflected waves can take place.

It can be concluded, then, from the above observations that if both end impedances are matched with the shaft characteristic impedance, i.e., $\overset{\#}{\Gamma}_0(0)=\overset{\#}{\Gamma}_n(\ell_n)\equiv \overset{\#}{0}$, or $\overset{\Delta}{z}_0(0)=\overset{\#}{z}_n(\ell_n)=\overset{\#}{C}_{21}\overset{\#}{C}{}_{11}^{-1}\equiv \overset{\#}{z}_s$, the effects of intermediate supports on the dynamic response of the rotating shaft are redundant. However, the presence of intermediate supports does raise the frequencies at which the critical speeds occur, including the fundamental frequency. Actually, one could place a predetermined number of intermediate supports

along the shaft sufficient to raise the fundamental frequency above the operating frequency of the shaft. Obviously, this would involve the acceptance of a penalty in added support weight.

Further inspection of Eq. (48) shows that even if the intermediate support conditions are not optimized, their effect is redundant. Thus, if end impedances can be matched to the characteristic impedance of the shaft, nothing is gained from the optimization of the intermediate supports in terms of minimum vibration control. Thus, one or two matched end supports should more than adequately suppress any excessive vibration response in hypercritical shafts of any length. The impedance values for the supports of the shaft, whether supported at one end or at both ends, are the same; the difference lies in the deflection amplitudes, where those of the single-ended shaft are twice those of the double-ended shaft.

The matching of end impedance with the shaft characteristic impedance is a formidable task, not only because impedances are a function of frequency but primarily because of the uniqueness of all the conditions required to control these impedances. Moreover, in practical applications, the choice in end support configurations is limited, since they usually are governed by such factors as transmission gears, couplings, and unwieldy mountings. Thus, the intermediate supports have to be employed for optimization purposes, since the matched end impedances are not available.

If all supports are optimized except those at both ends, i.e., if all reflection matrices are equal to null matrices except $\overset{\#}{\Gamma}_0(0)$, $\overset{\#}{\Gamma}_n(\ell_n) \neq \overset{\#}{0}$, the dynamic responses due to distributed driving forces, Eq. (27), may be expressed in the same form as Eq. (48) except for those pertaining to the 1^{st} and n^{th} spans. Those responses may be listed as follows:

On $\left[0, \ell_1\right]$, or the 1^{st} span,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\gg\!>}{\bar{U}}_1(x) \\ \overset{<\!\ll}{\bar{U}}_1(x) \end{bmatrix} \tag{49}$$

where

$$\overset{\gg\!>}{\bar{U}}_1(x) = \sum_{j=1}^{n} \overset{\gg\!>}{\bar{U}}_{1j}(x)$$

$$\overset{<\!\ll}{\bar{U}}_1(x) = \sum_{j=1}^{n} \overset{<\!\ll}{\bar{U}}_{1j}(x)$$

The expressions for $\overset{\ggg}{\bar{U}}_{1j}(x)$ and $\overset{\lll}{\bar{U}}_{1j}(x)$ can be obtained from Eq. (27) by letting i be 1 for j=i, and then letting j = n, n-1, . . . , 2 for j> 1. It should be noted that the expression for the reflected wave function $\overset{\ll}{\bar{U}}_{11}(x) \overset{\#}{=} 0$, and that the terms $\overset{>}{\bar{U}}_{(j-1)j}(x), \ldots, \overset{>}{\bar{U}}_{2j}(x)$ vanish also.

On $\left[\ell_{n-1}, \ell_n\right]$, or the n^{th} span,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\bar{U}}_n(x) \\ \\ \overset{\lll}{\bar{U}}_n(x) \end{bmatrix} \tag{50}$$

where

$$\overset{\ggg}{\bar{U}}_n(x) = \sum_{j=1}^{n} \overset{\ggg}{\bar{U}}_{nj}(x)$$

$$\overset{\lll}{\bar{U}}_n(x) = \sum_{j=1}^{n} \overset{\lll}{\bar{U}}_{nj}(x) \quad .$$

In this span, the expression for $\overset{\ggg}{\bar{U}}_{nj}(x)$ and $\overset{\lll}{\bar{U}}_{nj}(x)$ can be obtained from Eq. (27) by letting i be n for j=i, and then letting j be j = 1, 2, . . . , n-1 for j <n; also, the term $\overset{\gg}{\bar{U}}_{nn} \overset{\#}{=} 0$, and each term $\overset{<}{\bar{U}}_{(j+1)j}(x), \ldots, \overset{<}{\bar{U}}_{(n-1)j}(x) \overset{\#}{=} 0$.

It may be observed from the above expressions that only incident waves can exist in every interior span except for the two end spans, in which incident waves originated by driving forces are partially reflected at the end supports as indicated by the $\overset{\ggg}{\bar{U}}_{1j}(x)$, $\overset{\lll}{\bar{U}}_{1j}(x)$, $\overset{\ggg}{\bar{U}}_{nj}(x)$, and $\overset{\lll}{\bar{U}}_{nj}(x)$ terms.

The existence of partially reflected waves in both end spans is an unavoidable situation, since both left and right end supports are not optimized, i. e., are not matched with the shaft characteristic impedance. Thus, incident waves traveling from the interior spans to $x=\ell_1$ and $x=\ell_{n-1}$ simply pass on to $\left[0, \ell_1\right]$ and $\left[\ell_{n-1}, \ell_n\right]$, respectively, never to return to $\left[\ell_1, \ell_{n-1}\right]$ again. These waves will then propagate back and forth in the $\left[0, \ell_1\right]$ and $\left[\ell_{n-1}, \ell_n\right]$ intervals, to be damped out eventually

by the unmatched impedance at $x = 0-0$ or at $x = \ell_n+0$, respectively. It should be remembered from the first section of this chapter "Traveling Wave Concept" that for the reflected waves in the $[0, \ell_1]$ and $[\ell_{n-1}, \ell_n]$ intervals, the supports at $x = \ell_1$ and $x = \ell_{n-1}$ act as fixed supports which will reflect completely all incoming reflected waves, respectively, from the left and right. Thus, since reflected waves and modified waves vanish along the shaft, except for both end spans, the shaft system is optimized on $[\ell_1, \ell_{n-1}]$, but not on $[0, \ell_1]$ and $[\ell_{n-1}, \ell_n]$.

The minimization of vibration response in a shaft having end supports that do not lend themselves readily to optimization may be accomplished by letting the 1^{st} and the $(n-1)^{th}$ intermediate supports approach, respectively, the left and right ends of the shaft as closely as possible. In this case, the reflected waves will be restricted to very short end spans and, if all intermediate supports are matched with the shaft's impedance, only incident waves will exist in the $[\ell_1, \ell_{n-1}]$ portion of the shaft. Mathematically, the above statement is indicated in Eqs. (49) and (50), where as ℓ_1 and $(\ell_n-\ell_{n-1})$ approach zero, $\overset{\ggg}{\bar{U}}_{1j}(x)$, $\overset{\lll}{\bar{U}}_{1j}(x)$, $\overset{\ggg}{\bar{U}}_{nj}(x)$, and $\overset{\lll}{\bar{U}}_{nj}(x)$ also approach zero. It can be said, then, that the closer the matched 1^{st} and $(n-1)^{th}$ supports are placed to the left and right end supports, respectively, the more effective the amplitude suppression will be in the end portions, $[0, \ell_1]$ and $[\ell_{n-1}, \ell_n]$, of the shaft.

It has been observed earlier that matched or unmatched interior supports would be redundant if the end support impedances are matched to the characteristic impedance of the shaft. By the same reasoning, inspection of Eq. (48) should show also that for the case in which the two outermost interior supports are matched, all supports interior to these two supports are redundant. From all this discussion, the conclusion may be made that in the case for which the end supports are not available for optimization, one or two matched intermediate supports placed very close to the end supports should more than adequately suppress excessive vibration response for shafts of any length. The impedances of the interior supports should be the same for the shaft with one interior support placed close to one end and the shaft with two interior supports placed close to the ends. The only difference in behavior between these two shaft systems exists in the amplitude response. It is larger in the case of the shaft with one interior

support; how much larger this value of the amplitude response will be depends on the amount of wave reflection that will take place at the far end support. In the case of the shaft with two interior supports placed close to the end supports, no such reflection of incident waves will take place at either end, and the resulting deflecting amplitude response should be less than that for the single interior support case.

DETERMINATION OF THE MATCHED IMPEDANCES OF INTERIOR SUPPORTS

The optimum impedances, $\overset{\#}{Z}_j(\ell_j)$, $j = 1, 2, \ldots, n-1$, of interior supports are determined from the following conditions:

$$\left.\begin{aligned} &\overset{\#}{\Gamma}_{j\ell}(\overset{\#}{\ell_j})=0, \quad j = 1, 2, \ldots, n-1 \\ &\text{and} \\ &\overset{\#}{\Gamma}_{jr}(\overset{\#}{\ell_j})=0, \quad j = n-1, n-2, \ldots, 1 . \end{aligned}\right\} \tag{51}$$

It may be shown as before (see Eqs. (47)) that the above expressions are equivalent to setting

$$\left.\begin{aligned} &\overset{\triangle}{z}_{j\ell}(\ell_j)=\overset{\#}{C}_{21}\overset{\#}{C}{}^{-1}_{11}=\overset{\#}{z}_s , \quad j=1, 2, \ldots, n-1 \\ &\text{and} \\ &\overset{\#}{z}_{jr}(\ell_j)=\overset{\#}{C}_{21}\overset{\#}{C}{}^{-1}_{11}=\overset{\#}{z}_s , \quad j=n-1, n-2, \ldots, 1 . \end{aligned}\right\} \tag{52}$$

The details of $\overset{\triangle}{z}_{j\ell}(\ell_j)$ and $\overset{\#}{z}_{jr}(\ell_j)$ are summarized in Eqs. (17) through (24).

From these equations, a set of supports can be uniquely determined such that the shaft will have minimum dynamic response characteristics. However, it should be noted that the shaft's characteristic impedance is a function of frequency, ω ; hence, all matched support impedances must necessarily be functions of frequency. To provide these supports, in reality, with matched impedances is a formidable task, not only because they are a function of frequency, but primarily, as has been stated before in the preceding section, because of the uniqueness of the requirements for these impedances. Each support must be a translational and rotational mass-spring-damper unit with a built-in frequency dependency to meet these requirements. This is a physical situation that is not easily attained. Hence, from a practical point of view, the possibility of imposing weaker

conditions on support requirements for minimum vibration response of hypercritical shafts should be investigated.

QUASI-MATCHING

A compromise approach, "quasi-matching", to minimizing critical speed vibration may be realized in the form of weaker conditions on impedances at support locations. By examining the expressions of reflected and modified waves, some interesting observations may be made. First, by writing out the expression for the incident wave $\overset{>}{\bar{U}}_{ii}(x)$ in Eq. (27) in complete matrix notation,

$$\overset{>}{\bar{U}}_{ii}(x) = \sum_{k=1}^{k(i)} \begin{bmatrix} e^{-ie_1\sqrt{\omega}\,(x-a_{ik})} & 0 \\ 0 & e^{-e_2\sqrt{\omega}\,(x-a_{ik})} \end{bmatrix} C_+^{\#} C_f^{\#-1} \tilde{P}(a_{ik})$$

$$\times \left[H(x-a_{ik}) - H(x-\ell_i) \right]$$

The term

$$e^{-ie_1\sqrt{\omega}\,(x-a_{ik})} = \cos(e_1\sqrt{\omega}\,(x-a_{ik})) - i\ \sin(e_1\sqrt{\omega}\,(x-a_{ik}))$$

is a complex entry which affects only the phase relation of the wave as the argument $(x-a_{ik})$ increases. The other term, $e^{-e_2\sqrt{\omega}\,(x-a_{ik})}$, is a function for which the numerical value decays exponentially as $(x-a_{ik})$ increases. Generally, this term, $e^{-e_2\sqrt{\omega}\,(x-a_{ik})}$, is very small when compared to the first term, $e^{-ie_1\sqrt{\omega}\,(x-a_{ik})}$, which does not decay. Hence, it may be neglected with very little effect on the solution. Second, by examining the expression of reflected wave $\overset{\ll}{\bar{U}}_{ii}(x)$ in the i^{th} span as given by Eq. (27), the term, which corresponds to $\overset{\ll}{\bar{U}}_{ii}(x)\Big|_i$ (see Eqs. (33) and (35)), can also be written out in complete matrix notation:

$$
\overset{\#}{R}(\ell_i-x)\overset{\#}{\Gamma}_{ir}(\ell_i)\overset{>}{\bar{U}}_{ii}(\ell_i)
$$

$$
= \sum_{k=1}^{k(i)} \begin{bmatrix} (e^{-ie_1\sqrt{\omega}(\ell_i-a_{ik})}\Gamma_{ir}(\ell_i)_{11})e^{-ie_1\sqrt{\omega}(\ell_i-x)} & (e^{-e_2\sqrt{\omega}(\ell_i-a_{ik})}\Gamma_{ir}(\ell_i)_{12})e^{-ie_1\sqrt{\omega}(\ell_i-x)} \\ (e^{-ie_1\sqrt{\omega}(\ell_i-a_{ik})}\Gamma_{ir}(\ell_i)_{21})e^{-e_2\sqrt{\omega}(\ell_i-x)} & (e^{-e_2\sqrt{\omega}(\ell_i-a_{ik})}\Gamma_{ir}(\ell_i)_{22})e^{-e_2\sqrt{\omega}(\ell_i-x)} \end{bmatrix} \overset{\#}{C}_{+}\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{ik}) .
$$

In the reflection matrix $\overset{\#}{\Gamma}_{ir}(\ell_i)$, the only element not modified by an exponential decaying function is $\overset{\#}{\Gamma}_{ir}(\ell_i)_{11}$. Thus, for cases in which these elements become negligible with increasing ω, only the element which is not modified by an exponential decaying function need be considered. This leads to the simpler expression:

$$
\overset{\#}{R}(\ell_i-x)\overset{\#}{\Gamma}_{ir}(\ell_i)\overset{>}{\bar{U}}_{ii}(\ell_i)
$$

$$
= \sum^{k(i)} \begin{bmatrix} (e^{-ie_1\sqrt{\omega}(\ell_i-a_{ik})}\ \Gamma_{ir}(\ell_i)_{11})e^{-ie_1\sqrt{\omega}(\ell_i-x)} & 0 \\ 0 & 0 \end{bmatrix} \overset{\#}{C}_{+}\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{ik})
$$

The same manipulation may be applied to all the other reflected and modified waves. Hence, as a logical conclusion, the shaft matching conditions as stated in Eqs. (51) can be replaced by weaker ones as follows:

$$
\left.\begin{aligned}
&\Gamma_{1\ell}(\ell_1)_{11}=\Gamma_{2\ell}(\ell_2)_{11}= \ldots = \Gamma_{(n-1)\ell}(\ell_{n-1})_{11}=0 \\
&\text{and} \\
&\Gamma_{(n-1)r}(\ell_{n-1})_{11}=\Gamma_{(n-2)r}(\ell_{n-2})_{11}= \ldots = \Gamma_{1r}(\ell_1)=0
\end{aligned}\right\} \qquad (53)
$$

where the subscript "11" indicates the element of the first row and the first column of each reflection matrix. If all interior supports are designed such that the above conditions are satisfied, then the shaft is termed quasi-matched. By examining Eqs. (23) and (24) and their related

expressions, it is evident that under the condition of quasi-matching, each term $\overset{\Delta}{z}_{1\ell}(\ell_1)$, $\overset{\Delta}{z}_{2\ell}(\ell_2)$, . . . , $\overset{\Delta}{z}_{(n-1)\ell}(\ell_{n-1})$, $\overset{\#}{z}_{(n-1)r}(\ell_{n-1})$, $\overset{\#}{z}_{(n-2)r}(\ell_{n-2})$, . . . , $\overset{\#}{z}_{ir}(\ell_1)$, approaches $\overset{\#}{z}_s = \overset{\#}{C}_{21}\overset{\#}{C}_{11}^{-1}$(Eqs. (47)), which relates to shaft characteristic impedance (i. e., matched conditions) as $\exp(-e_2\sqrt{\omega}\,(\ell_i-a_{ik}))$ or $\exp(-e_2\sqrt{\omega}\,(\ell_i-x))$, $i = 1, 2, \ldots, n$, tends to zero, where a_{ik} or x is on $\left[0, \ell_n\right]$.

The application of the criteria given in Eqs. (53) for the quasi-matching of interior support impedances to the characteristic impedance of the shaft to the design of several support configurations will be illustrated in Chapter 4. For convenience sake, a shaft with only one interior support and two end supports will be studied.

expressions, it is evident that under the condition of quasi-matching, each term $\overset{\Delta}{z}_{1\ell}(\ell_1)$, $\overset{\Delta}{z}_{2\ell}(\ell_2)$, . . . , $\overset{\Delta}{z}_{(n-1)\ell}(\ell_{n-1})$, $\overset{\#}{z}_{(n-1)r}(\ell_{n-1})$, $\overset{\#}{z}_{(n-2)r}(\ell_{n-2})$, . . . , $\overset{\#}{z}_{ir}(\ell_1)$, approaches $\overset{\#}{z}_s = \overset{\#}{C}_{21}\overset{\#}{C}_{11}^{-1}$(Eqs. (47)), which relates to shaft characteristic impedance (i.e., matched conditions) as $\exp(-e_2\sqrt{\omega}\,(\ell_i - a_{ik}))$ or $\exp(-e_2\sqrt{\omega}\,(\ell_i - x))$, $i = 1, 2, \ldots, n$, tends to zero, where a_{ik} or x is on $[0, \ell_n]$.

The application of the criteria given in Eqs. (53) for the quasi-matching of interior support impedances to the characteristic impedance of the shaft to the design of several support configurations will be illustrated in Chapter 4. For convenience sake, a shaft with only one interior support and two end supports will be studied.

CHAPTER IV

TRANSMISSION LINE ANALOGY SOLUTION OF THE SUPERCRITICAL SHAFT WITH ONE INTERMEDIATE SUPPORT--USED AS AN ILLUSTRATION PROBLEM FOR QUASI-MATCHING

DESCRIPTION OF PROBLEM

A uniform rotating shaft with one interior support will be used in this chapter for the development of equations for different support configurations in terms of quasi-matched conditions. These expressions should be useful to the designer in that impedance values termed "best" for the supports may be established directly without an excessive amount of computations.

The criterion for quasi-matching of the interior support impedance with the shaft impedance, looking to the left and right of $x = \ell_1$ is, referring to Eqs. (53),

$$\left.\begin{aligned} &\Gamma_{1\ell}(\ell_1)_{11}=0 \\ &\Gamma_{1r}(\ell_1)_{11}=0 \end{aligned}\right\} \qquad (54)$$

In general, $\Gamma_{1\ell}(\ell_1)_{11} \neq \Gamma_{1r}(\ell_1)_{11}$, or $\overset{\#}{\Gamma}_{1\ell}(\ell_1) \neq \overset{\#}{\Gamma}_{1r}(\ell_1)$; the condition $\overset{\#}{\Gamma}_{1\ell}(\ell_1)=\overset{\#}{\Gamma}_{1r}(\ell_1)$ exists only if the interior support is placed at the midpoint ($\ell_1 = 1/2\ \ell_2$) of the shaft span, and both end supports have the same configuration.

A physical model of the shaft used for the analysis in this chapter is shown in Figure 16. This corresponds to the general case shown in Figure 4 by letting $n = 2$. If a shaft is used here having identical end support configurations,

$$\overset{\Delta}{Z}_0(0)=\overset{\#}{Z}_2(\ell_2) \quad . \qquad (55)$$

By direct substitution of Eq. (55) into the expressions of $\overset{\#}{\Gamma}_{1\ell}(\ell_1)$ and $\overset{\#}{\Gamma}_{1r}(\ell_1)$ and assuming that $\ell_1 = 1/2\ \ell_2$, it can be shown that (see Appendix H)

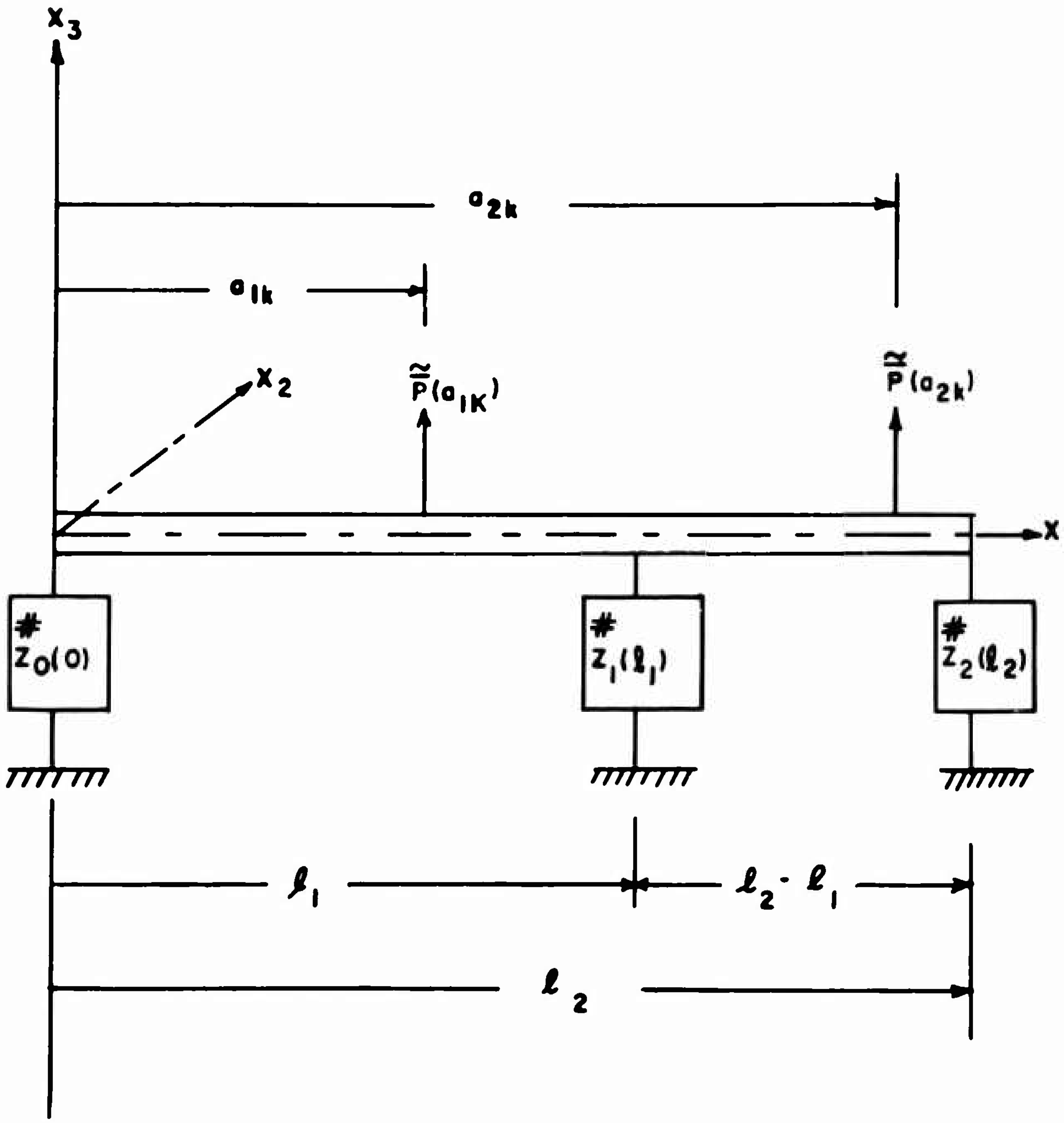

Figure 16. Uniform Shaft With One Interior Support at the Midpoint of Total Span.

$$
\left.\begin{aligned}
&\overset{\#}{\Gamma}_{1\ell}(\ell_1) \equiv \overset{\#}{\Gamma}_{1r}(\ell_1)\\
&\Gamma_{1\ell}(\ell_1)_{11} \equiv \Gamma_{1r}(\ell_1)_{11}\ .
\end{aligned}\right\} \qquad (56)
$$

The conditions of Eqs. (55) and (56) will be used in the development of a number of example problems of this chapter.

REFLECTION MATRICES AND SUPPORT IMPEDANCES

Reflection Matrices at Supports in Terms of Impedances

A set of corresponding equations for the shaft system shown in Figure 16 ($\ell_1 \neq 1/2\ \ell_2$) can be obtained from Chapter 2 simply by letting $n = 2$ in Eqs. (15)-(22), as follows:

$$
\left.\begin{aligned}
&\overset{\#}{\Gamma}_0(0) = \left[-\overset{\#}{C}_{22} + \overset{\triangle}{z}_0(0)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21} - \overset{\triangle}{z}_0(0)\overset{\#}{C}_{11}\right]\\
&\overset{\#}{\Gamma}_{1\ell}(\ell_1) = \left[-\overset{\#}{C}_{22} + \overset{\triangle}{z}_{1\ell}(\ell_1)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21} - \overset{\triangle}{z}_{1\ell}(\ell_1)\overset{\#}{C}_{11}\right]\\
&\overset{\#}{\Gamma}_{1r}(\ell_1) = \left[-\overset{\#}{C}_{22} + \overset{\#}{z}_{1r}(\ell_1)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21} - \overset{\#}{z}_{1r}(\ell_1)\overset{\#}{C}_{11}\right]\\
&\overset{\#}{\Gamma}_2(\ell_2) = \left[-\overset{\#}{C}_{22} + \overset{\#}{z}_2(\ell_2)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21} - \overset{\#}{z}_2(\ell_2)\overset{\#}{C}_{11}\right]
\end{aligned}\right\} \qquad (57)
$$

where

$$
\left.\begin{aligned}
&\overset{\triangle}{z}_{1\ell}(\ell_1) = \overset{\triangle}{z}_1(\ell_1) + \overset{\triangle}{z}_{0\ell}(\ell_1)\\
&\overset{\#}{z}_{1r}(\ell_1) = \overset{\#}{z}_1(\ell_1) + \overset{\#}{z}_{2r}(\ell_1)
\end{aligned}\right\}
$$

$$
\left.\begin{aligned}
&\overset{\triangle}{z}_{0\ell}(\ell_1) = \left[\overset{\#}{C}_{21} + \overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(\ell_1)\right]\left[\overset{\#}{C}_{11} + \overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(\ell_1)\right]^{-1}\\
&\overset{\#}{z}_{2r}(\ell_1) = \left[\overset{\#}{C}_{21} + \overset{\#}{C}_{22}\overset{\#}{\Gamma}_2(\ell_1)\right]\left[\overset{\#}{C}_{11} + \overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]^{-1}
\end{aligned}\right\}
$$

$$
\left.\begin{aligned}
&\overset{\#}{\Gamma}_0(\ell_1) = \overset{\#}{R}(\ell_1)\overset{\#}{\Gamma}_0(0)\overset{\#}{R}(\ell_1)\\
&\overset{\#}{\Gamma}_2(\ell_1) = \overset{\#}{R}(\ell_2 - \ell_1)\overset{\#}{\Gamma}_2(\ell_2)\overset{\#}{R}(\ell_2 - \ell_1)\ .
\end{aligned}\right\}
$$

Impedances in Terms of Reflection Matrices

A set of corresponding equations for the shaft system shown in Figure 16 ($\ell_1 \neq 1/2\,\ell_2$) can be obtained from Chapter 2 simply by letting n = 2 in Eqs. (23) and (24), as follows:

$$\left.\begin{aligned}
&\text{On } [0,\ \ell_1], \\
&\quad \overset{\triangle}{z}_{0\ell}(x) = \left[\overset{\#}{C}_{21} + \overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(x)\right]\left[\overset{\#}{C}_{11} + \overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(x)\right]^{-1} \\
&\text{On } [\ell_1,\ \ell_2], \\
&\quad \overset{\triangle}{z}_{1\ell}(x) = \left[\overset{\#}{C}_{21} + \overset{\#}{C}_{22}\overset{\#}{\Gamma}_{1\ell}(x)\right]\left[\overset{\#}{C}_{11} + \overset{\#}{C}_{12}\Gamma_{1\ell}(x)\right]^{-1} \\
&\text{On } [\ell_1,\ \ell_2], \\
&\quad \overset{\#}{z}_{2r}(x) = \left[\overset{\#}{C}_{21} + \overset{\#}{C}_{22}\overset{\#}{\Gamma}_2(x)\right]\left[\overset{\#}{C}_{11} + \overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(x)\right]^{-1} \\
&\text{On } [0,\ \ell_1], \\
&\quad \overset{\#}{z}_{1r}(x) = \left[\overset{\#}{C}_{21} + \overset{\#}{C}_{22}\Gamma_{1r}(x)\right]\left[\overset{\#}{C}_{11} + \overset{\#}{C}_{12}\Gamma_{1r}(x)\right]^{-1}
\end{aligned}\right\} \quad (58)$$

SOLUTION IN WAVE FORM

From Eq. (27) in Chapter 2, the corresponding solution in traveling wave form for the shaft system shown in Figure 16 ($\ell_1 \neq 1/2\,\ell_2$) can be obtained as follows:

$$\left.\begin{aligned}
&\text{On } [0,\ \ell_1], \\
&\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\bar{U}_1(x)} \\ \overset{\lll}{\bar{U}_1(x)} \end{bmatrix} \\
&\text{On } [\ell_1,\ \ell_2], \\
&\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\bar{U}_2(x)} \\ \overset{\lll}{\bar{U}_2(x)} \end{bmatrix}
\end{aligned}\right\} \quad (59)$$

where

$$
\left.\begin{aligned}
\overset{>>>}{\bar{U}}_1(x) &= \overset{>>>}{\bar{U}}_{11}(x)+\overset{>>>}{\bar{U}}_{12}(x) \\
\overset{<<<}{\bar{U}}_1(x) &= \overset{<<<}{\bar{U}}_{11}(x)+\overset{<<<}{\bar{U}}_{12}(x)
\end{aligned}\right\}
$$

$$
\left.\begin{aligned}
\overset{>>>}{\bar{U}}_2(x) &= \overset{>>>}{\bar{U}}_{21}(x)+\overset{>>>}{\bar{U}}_{22}(x) \\
\overset{<<<}{\bar{U}}_2(x) &= \overset{<<<}{\bar{U}}_{21}(x)+\overset{<<<}{\bar{U}}_{22}(x)
\end{aligned}\right\}
$$

$$
\left.\begin{aligned}
&\overset{>>>}{\bar{U}}_{11}(x)=\overset{>>}{\bar{U}}_{11}(x)+\overset{>}{\bar{U}}_{11}(x) \\
&\overset{<<<}{\bar{U}}_{11}(x)=\overset{<<}{\bar{U}}_{11}(x)+\overset{<}{\bar{U}}_{11}(x) \\
&\overset{>}{\bar{U}}_{11}(x)=\sum_{k=1}^{k(1)} \overset{\#}{R}(x-a_{1k})\overset{\#}{C}_+\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{1k})\left[H(x-a_{1k})-H(x-\ell_1)\right] \\
&\overset{>}{\bar{U}}_{11}(x)=\sum_{k=1}^{k(1)} \overset{\#}{R}(a_{1k}-x)\overset{\#}{C}_-\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{1k})H(a_{1k}-x) \\
&\overset{<<}{\bar{U}}_{11}(x)=\overset{\#}{R}(x)\left[I-\overset{\#}{\Gamma}_0(0)\overset{\#}{\Gamma}_{1r}(0)\right]^{-1}\overset{\#}{\Gamma}_0(0)\left[\overset{\#}{R}(\ell_1)\overset{\#}{\Gamma}_{1r}(\ell_1)\overset{>}{\bar{U}}_{11}(\ell_1)+\overset{<}{\bar{U}}_{11}(0)\right] \\
&\qquad\qquad \times H(\ell_1-x) \\
&\overset{<<}{\bar{U}}_{11}(x)=\overset{\#}{R}(\ell_1-x)\left[I-\overset{\#}{\Gamma}_{1r}(\ell_1)\overset{\#}{\Gamma}_0(\ell_1)\right]^{-1}\overset{\#}{\Gamma}_{1r} \\
&\qquad\qquad \times\left[\overset{>}{\bar{U}}_{11}(\ell_1)+\overset{\#}{R}(\ell_1)\overset{\#}{\Gamma}_0(0)\overset{<}{\bar{U}}_{11}(0)\right]H(\ell_1-x)
\end{aligned}\right\}
$$

$$\overset{\ggg}{\bar U}_{22}(x)=\overset{\gg}{\bar U}_{22}(x)+\overset{>}{\bar U}_{22}(x)$$

$$\overset{\lll}{\bar U}_{22}(x)=\overset{\ll}{\bar U}_{22}(x)+\overset{<}{\bar U}_{22}(x)$$

$$\overset{>}{\bar U}_{22}(x)=\sum_{k=1}^{k(2)}\overset{\#}{R}(x-a_{2k})\overset{\#}{C}_{+}\overset{\#}{C}_{f}^{-1}\overset{\simeq}{P}(a_{2k})H(x-a_{2k})$$

$$\overset{<}{\bar U}_{22}(x)=\sum_{k=1}^{k(2)}\overset{\#}{R}(a_{2k}-x)\overset{\#}{C}_{-}\overset{\#}{C}_{f}^{-1}\overset{\simeq}{P}(a_{2k})\left[H(a_{2k}-x)-H(\ell_1-x)\right]$$

$$\overset{\gg}{\bar U}_{22}(x)=\overset{\#}{R}(x-\ell_1)\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{1\ell}(\ell_1)\overset{\#}{\Gamma}_{2}(\ell_1)\right]^{-1}\overset{\#}{\Gamma}_{1\ell}(\ell_1)$$

$$\times\left[\overset{\#}{R}(\ell_2-\ell_1)\overset{\#}{\Gamma}_{2}(\ell_1)\overset{>}{\bar U}_{22}(\ell_2)+\overset{<}{\bar U}_{22}(\ell_1)\right]H(x-\ell_1)$$

$$\overset{\ll}{\bar U}_{22}(x)=\overset{\#}{R}(\ell_2-x)\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{2}(\ell_2)\overset{\#}{\Gamma}_{1\ell}(\ell_2)\right]^{-1}\overset{\#}{\Gamma}_{2}(\ell_2)$$

$$\times\left[\overset{>}{\bar U}_{22}(\ell_2)+\overset{\#}{R}(\ell_2-\ell_1)\overset{\#}{\Gamma}_{1\ell}(\ell_1)\overset{<}{\bar U}_{22}(\ell_2)\right]H(x-\ell_1)$$

$$\overset{\ggg}{\bar U}_{12}(x)=\overset{\#}{R}(x)\overset{\#}{\Gamma}_{0}(0)\overset{\#}{R}(\ell_1)\left[\overset{\#}{I}+\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{0}(\ell_1)\right]^{-1}\left[\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\bar U}_{22}(\ell_1)+\overset{\lll}{\bar U}_{22}(\ell_1)\right]$$

$$\times H(\ell_1-x)$$

$$\overset{\lll}{\bar U}_{12}(x)=\overset{\#}{R}(\ell_1-x)\left[\overset{\#}{I}+\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{0}(\ell_1)\right]^{-1}\left[\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\bar U}_{22}(\ell_1)+\overset{\lll}{\bar U}_{22}(\ell_1)\right]H(\ell_1-x)$$

$$\overset{\ggg}{\bar U}_{21}(x)=\overset{\#}{R}(x-\ell_1)\left[\overset{\#}{I}+\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{2}(\ell_1)\right]^{-1}\left[\overset{\ggg}{\bar U}_{11}(\ell_1)+\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\bar U}_{11}(\ell_1)\right]H(x-\ell_1)$$

$$\overset{\lll}{\bar U}_{21}(x)=\overset{\#}{R}(\ell_2-x)\overset{\#}{\Gamma}_{2}(\ell_2)\overset{\#}{R}(\ell_2-\ell_1)\left[\overset{\#}{I}+\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{2}(\ell_1)\right]^{-1}$$

$$\times\left[\overset{\ggg}{\bar U}_{11}(\ell_1)+\overset{\#}{C}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\bar U}_{11}(\ell_1)\right]H(x-\ell_1)\quad .$$

The preceding Eqs. (57), (58), and (59) give the pertinent formulas required for the optimization of support conditions of the two-span shaft with the interior support at position ℓ_1 and two general support configurations at the ends. Also from Eqs. (59), the dynamic response, i.e., the induced deflections and forces due to distributed driving forces, of the shaft under investigation can be calculated directly.

DETERMINATION OF THE QUASI-MATCHED IMPEDANCE AT THE INTERIOR SUPPORT

In the subsequent set of example problems for the shaft on three supports it is assumed, for convenience sake, that both end supports have the same configuration so that Eq. (55) holds. In addition, the interior support is assumed to act at the mid-span ($\ell_1 = 1/2\ \ell_2$) of the shaft such that

$$\overset{\#}{\Gamma}_{1\ell}(\ell_1) = \overset{\#}{\Gamma}_{1r}(\ell_1),$$

or in terms of the quasi-matching condition for the impedance of that support to the characteristic impedance of the shaft,

$$\Gamma_{1\ell}(\ell_1)_{11} \equiv \Gamma_{1r}(\ell_1)_{11} = 0 \quad . \tag{60}$$

Since $\overset{\#}{\Gamma}_{1\ell}(\ell_1) \equiv \overset{\#}{\Gamma}_{1r}(\ell_1)$, only the condition $\overset{\#}{\Gamma}_{1r}(\ell_1) = 0$ needs to be considered in the following analysis to determine design criteria for the interior support meeting the requirements for quasi-matching. Thus the proper values will be obtained for both $\overset{\#}{Z}_{1r}(\ell_1)$ and $\overset{\Delta}{Z}_{1\ell}(\ell_1)$ which are the impedance quantities the interior support must provide, respectively, looking to the right and left of the support for a minimum vibration response of the symmetrical shaft.

The matrix forms for the impedance at each support are as follows (see Appendix H for explanations):

$$\left.\begin{array}{ll}
\text{At } x=0, & \overset{\#}{Z}_0(0) = \begin{bmatrix} Z_{0(11)} & -Z_{0(12)} \\ -Z_{0(21)} & Z_{0(22)} \end{bmatrix} \\
\text{At } x=\ell_1, & \overset{\#}{Z}_1(\ell_1) = \begin{bmatrix} Z_{1(11)} & 0 \\ 0 & Z_{1(22)} \end{bmatrix}
\end{array}\right\} \tag{61}$$

At $x = \ell_2$, $$\overset{\#}{Z}_2(\ell_2) = \begin{bmatrix} Z_{2(11)} & Z_{2(12)} \\ \\ Z_{2(21)} & Z_{2(22)} \end{bmatrix}$$

where $\overset{\Delta}{Z}_0(0) \equiv \overset{\#}{Z}_2(\ell_2)$, i. e., $Z_{0(ij)}(0) = Z_{2(ij)}(\ell_2)$, i, j=1, 2.

If the following equation,

$$\overset{\#}{\Gamma}_{1r}(\ell_1) = \left[-\overset{\#}{C}_{22} + \overset{\#}{z}_{1r}(\ell_1)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21} - \overset{\#}{z}_{1r}(\ell_1)\overset{\#}{C}_{11}\right] ,$$

and its related expressions (see Eqs. (57) and (58)) are used, then $\Gamma_{1r}(\ell_1)_{11} = 0$ leads to the following expression which corresponds to a quasi-matched interior support (see the mathematical derivations in Appendix I):

$$(1+e_3^2)(z_{1r(22)} + ie_1e_2z_{1r(11)}) + (e_1e_3^2 + ie_2)(z_{1r(12)}z_{1r(21)} - z_{1r(11)}z_{1r(22)})$$

$$-e_3(e_2 + ie_1)(z_{1r(12)} + z_{1r(21)}) - (e_1 + ie_2e_3^2) = 0 \tag{62}$$

where

$$z_{1r}(\ell_1) = \begin{bmatrix} z_{1r(11)} & z_{1r(12)} \\ \\ z_{1r(21)} & z_{1r(22)} \end{bmatrix}$$

$$z_{1r(11)} = z_{1(11)} + \frac{(1+e_3^2)}{\det\left[\overset{\#}{C}_{11} + \overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]}\left[\Gamma_{2(11)}(\ell_2)e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)} - 1\right]$$

$$z_{1r(12)} = z_{1r(21)} = \frac{e_3}{\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]}$$

$$\times\left[-(e_2+ie_1)\Gamma_{2(11)}(\ell_2)e^{-2ie\sqrt{\omega}(\ell_2-\ell_1)}+(e_2-ie_1)\right]$$

$$z_{1r(22)} = z_{1(22)}+\frac{ie_1e_2(1+e_3^2)}{\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]}\left[\Gamma_{2(11)}(\ell_2)e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}+1\right]$$

$$\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right] = -e_1e_3^2\left[e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_{2(11)}(\ell_2)+1\right]-$$

$$ie_2\left[e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_{2(11)}(\ell_2)-1\right]$$

$$\Gamma_{2(11)}(\ell_2) = \frac{1}{\det\left[-\overset{\#}{C}_{22}+\overset{\#}{z}_2(\ell_2)\overset{\#}{C}_{12}\right]}\Big[(1+e_3^2)(z_{2(22)}+ie_1e_2z_{2(11)})+$$

$$(e_1e_3^2+ie_2)(z_{2(12)}z_{2(21)}-z_{2(11)}z_{2(22)})-$$

$$e_3(e_2+ie_1)(z_{2(12)}+z_{2(21)})-(e_1+ie_2e_3^2)\Big]$$

$$\det\left[-\overset{\#}{C}_{22}+\overset{\#}{z}_2(\ell_2)\overset{\#}{C}_{12}\right]$$

$$= \cdot(ie_3+ie_1e_3z_{2(11)}+z_{2(12)})(e_2e_3-ie_2z_{2(21)}+ie_3z_{2(22)})+$$

$$(e_1+ie_1e_3z_{2(21)}+z_{2(22)})(1-ie_2z_{2(11)}+ie_3z_{2(12)}) \quad .$$

It appears from the preceding discussion and equations that the example problems have been severely limited to several simple cases of the symmetrically supported shaft. That this is not so, however, will become apparent, if the following principles are understood:

Consider the case

$$\ell_1 \neq \ell_2$$

then,

$$\overset{\#}{z}_{1r}(\ell_1) \neq \overset{\Delta}{z}_1(\ell_1) \; ;$$

this means that besides establishing $\overset{\#}{z}_{1r}(\ell_1)$, a second set of conditions for $\overset{\Delta}{z}_{1r}(\ell_1)$ must be met by the interior support. Equations for evaluating $\overset{\Delta}{z}_{1\ell}(\ell)$ may be developed in accordance with the steps indicated in Appendix I. To control the shaft's minimum vibration response completely, it is necessary that the interior support consists of two sets of impedances, each optimized for its own set of standing waves: $\overset{\#}{z}_{1r}(\ell_1)$ for the waves traveling to the right and originating in the $[0, \ell_1]$ span, and $\overset{\Delta}{z}_{1\ell}(\ell_1)$ for the waves traveling to the left and originating in the $[\ell_1, \ell_2]$ span. As has been suggested in the preceding chapter, page 58 , optimized interior supports act as one-way filters in that they let incoming incident waves through, but block the reflected waves approaching from the other side, except in the case when the support exhibits impedance characteristics which satisfy conditions for incoming waves from both the left and right sides. In general, it is not easy to build supports meeting such conditions. Therefore, it is suggested that the interior support be placed as close to an end support as possible so that it, in effect, replaces that support. In the example problems of this chapter it would mean placing the interior support in close proximity of the right end support since Eq. (62) is given in terms of the components of $\overset{\#}{z}_{1r}(\ell_1)$.

It follows from these arguments that Eq. (62) can be applied also to problems in which the interior support is not placed at midspan, but placed in close proximity of the right-hand end support. In this case, $\ell_1 \approx \ell_2$, and inspection of Eq. (62) shows that the effect of the right end support is essentially eliminated since the term $\Gamma_{2(11)}(\ell_2)$ must vanish for $\ell_1 = \ell_2$. The terms in Eq. (62) simplify to:

$$z_{1r(11)} = z_{1(11)} - \frac{(1+e_3^2)}{-e_1 e_3^2 + ie_2}$$

$$z_{1r(12)} = z_{1r(21)} = \frac{e_3(e_2 - ie_1)}{-e_1 e_3^2 + ie_2}$$

$$z_{1r(22)} = z_{1(22)} + \frac{ie_1 e_2(1+e_3^2)}{-e_1 e_3^2 + ie_2}$$

which together with Eq. (62) are the equations which Nelson derived for single span shafts (reference 14). Thus, the results for the symmetry cases of the shaft can be used for or extended to the problems in which the interior support essentially replaces one of the end supports. It should be noted here that Eq. (62) and associated expressions for $z_{1r(11)}$, $z_{1r(12)}$, $z_{1r(21)}$, and $z_{1r(22)}$ could be worked into relatively simple design formulas in terms of specific support conditions, such as translational supports or rotational supports, especially in cases for which the shear deformation, rotational inertia, and gyroscopic terms are neglected; i. e., when $e_1 = e_2 = e_3 = 1$.

EXAMPLES

To illustrate the existence of many quasi-matched interior support impedances yielding a "best" vibration response behavior, and to show the manner in which the impedance at each support is calculated, some specific examples will be studied.

Three-Support Shaft System with Specified Interior Support Configuration

As a first example, consider a shaft with both ends having the same configuration, and with its interior support consisting of both a translational and a rotational spring and damper unit of negligible mass as indicated by K_{1T}, K_{1R}, C_{1T}, C_{1R}. This shaft system is shown in Figure 17. Consider an infinitesimal shaft element at $x = \ell_1$; all forces acting on this element are also shown in Figure 17.

Since, from Figure 17, all forces are assumed to be in the positive sense, the following force equilibrium condition exists at $x = \ell_1$ as

$$\overline{F}(\ell_1 - 0) - \overline{F}(\ell_1 + 0) = -\overline{R}(\ell_1)$$

and

$$-R(\ell_1) = \begin{bmatrix} C_{1T}Y_{1t}(\ell_1) + K_{1T}Y_1(\ell_1) \\ \\ C_{1R}Y_{2t}(\ell_1) + K_{1R}Y_2(\ell_1) \end{bmatrix} .$$

If the above relationships are restated in Laplace transform form,

$$\overset{\simeq}{F}(\ell_1 - 0) - \overset{\simeq}{F}(\ell_1 + 0) = -\overset{\simeq}{R}(\ell_1)$$

and

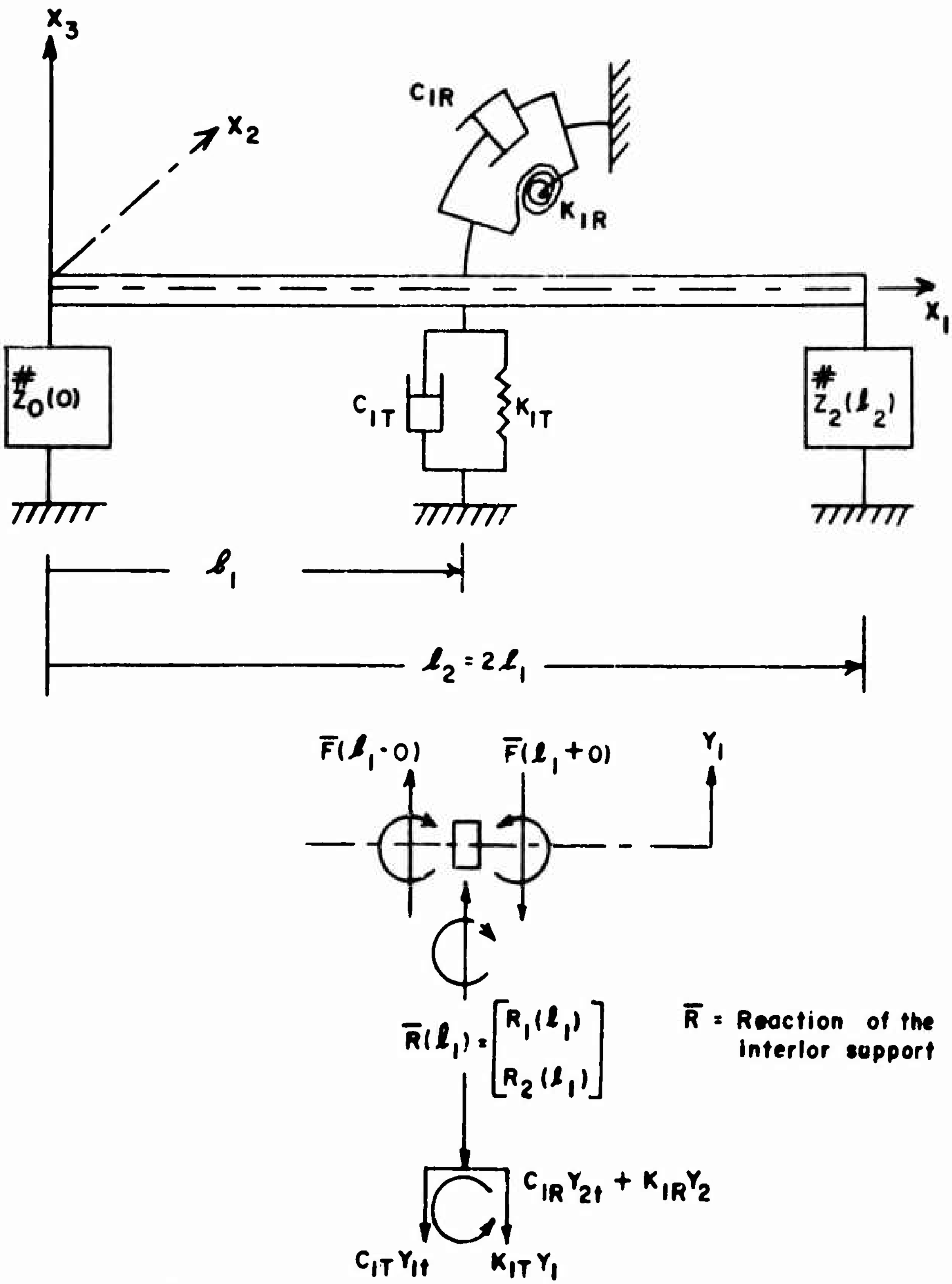

Figure 17. Three-support Shaft System With Specified Interior Support Configuration.

$$
-\tilde{R}(\ell_1) = s \begin{bmatrix} C_{1T} + \frac{1}{s} K_{1T} & 0 \\ 0 & C_{1R} + \frac{1}{s} K_{1R} \end{bmatrix} \tilde{Y}(\ell_1) \quad .
$$

If this expression is substituted back into the first relationship for the force equilibrium conditions at $x = \ell_1$ and if this expression is compared with Eq. (13), which may be written as

$$
\tilde{F}(\ell_1 - 0) - \tilde{F}(\ell_1 + 0) = s \overset{\#}{Z}_1(\ell_1) \tilde{Y}(\ell_1) \quad ,
$$

it is concluded that

$$
\overset{\#}{Z}_1(\ell_1) = \begin{bmatrix} C_{1T} + \frac{1}{s} K_{1T} & 0 \\ 0 & C_{1R} + \frac{1}{s} K_{1R} \end{bmatrix} , \tag{63}
$$

which is the interior support impedance. If Eq. (61) is used and if Eq. (62) is satisfied, the values for K_{1T}, K_{1R}, C_{1T}, and C_{1R} can be obtained; the "best" vibration response possible for the shaft supported by the type of supports shown in Figure 17 will result, provided that the impedance values of the end supports are known.

<u>Three-Support Shaft System with Floating Ring Damper Assembly as the Interior Support</u>

A second and more complicated configuration of the interior support is shown in Figure 18. In this case, the impedance of the interior support is as follows (see the mathematical derivations in Appendix J):

$$
\overset{\#}{Z}_1(\ell_1) = \begin{bmatrix} \dfrac{s^4 M_a M_b C + s^3 M_a M_b (K_b + K_c) + s[(M_a + M_b) K_a (K_b + K_c) + M_a K_b K_c]}{s^3 M_b C + s^2 M_b (K_b + K_c) + s(K_a + K_c) C + (K_a K_b + K_b K_c + K_c K_a)} & \\ 0 & \end{bmatrix}
$$

$$
\begin{bmatrix} \dfrac{+ s[(M_a + M_b) K_a (K_b + K_c) + M_a K_b K_c] + K_a K_c C + \frac{1}{s} K_a K_b K_c}{} & 0 \\ & 0 \end{bmatrix} .
$$

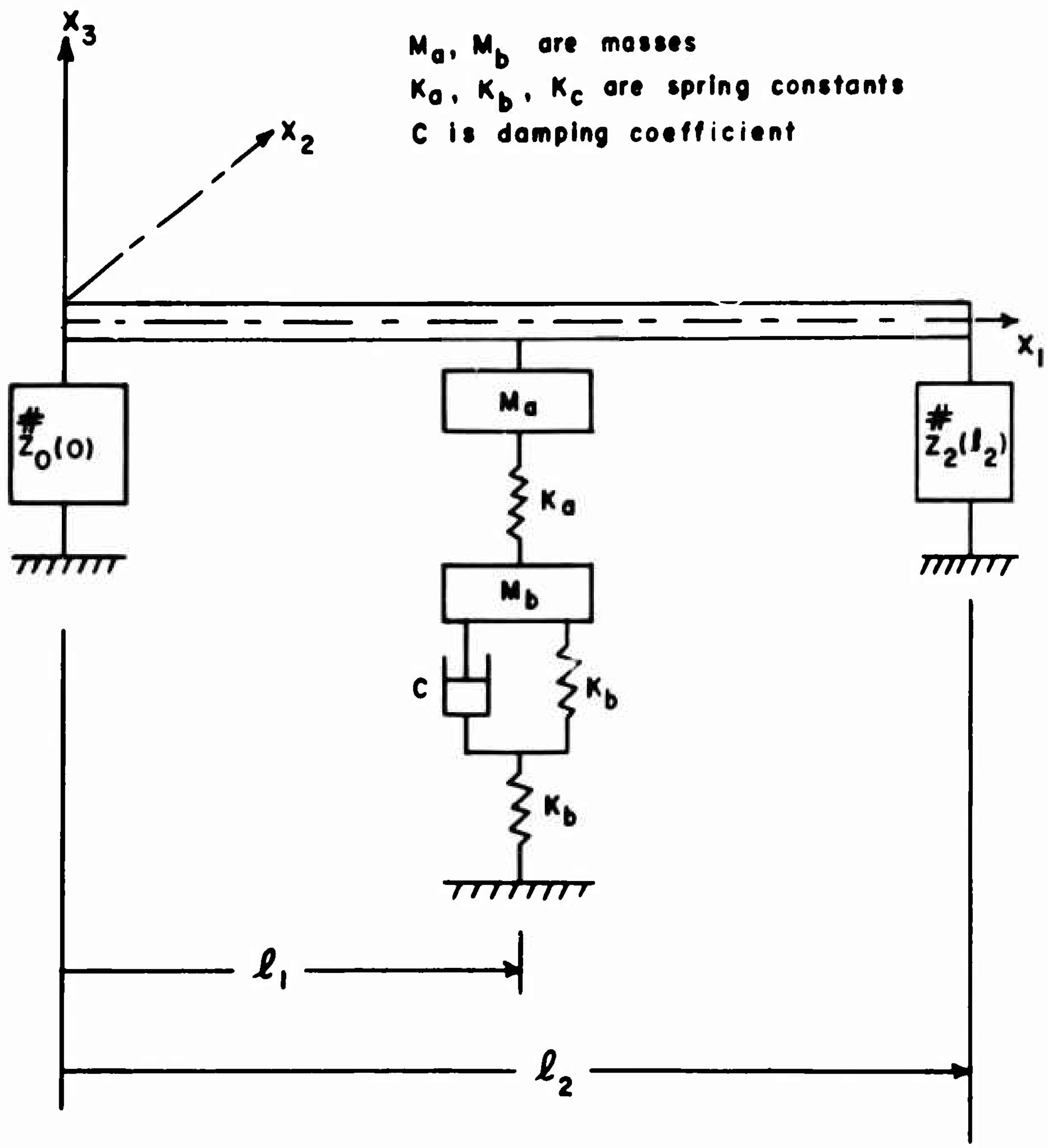

Figure 18. Three-support Shaft System With Floating Ring Damper Assembly as the Interior Support.

This rather general configuration can be used to represent a wide variety of actual support configurations. In this case, it was selected to serve as an idealization of the floating ring damper bearing assembly (reference 8). Again, Eqs. (61) and (62) should be used to establish the "best" values for M_a, M_b, K_a, K_b, K_c and C.

<u>Three-Support Shaft System with Specified Support Configurations</u>

A third and more specific shaft system is the one shown in Figure 19. For simplicity, all supports are assumed to have the same configuration. The support impedances which can be derived in the manner shown in Appendix I can be expressed as follows (see Eq. (63)):

$$\left.\begin{aligned}
\overset{\#}{Z}_0(o) &= \begin{bmatrix} Z_{0(11)} & 0 \\ 0 & Z_{0(22)} \end{bmatrix} = \begin{bmatrix} C_{0T}+\frac{1}{s}K_{0T} & 0 \\ 0 & C_{0R}+\frac{1}{s}K_{0R} \end{bmatrix} \\
\overset{\#}{Z}_1(\ell_1) &= \begin{bmatrix} Z_{1(11)} & 0 \\ 0 & Z_{1(22)} \end{bmatrix} = \begin{bmatrix} C_{1T}+\frac{1}{s}K_{1T} & 0 \\ 0 & C_{1R}+\frac{1}{s}K_{1R} \end{bmatrix} \\
\overset{\#}{Z}_2(\ell_2) &= \begin{bmatrix} Z_{2(11)} & 0 \\ 0 & Z_{2(22)} \end{bmatrix} = \begin{bmatrix} C_{2T}+\frac{1}{s}K_{2T} & 0 \\ 0 & C_{2R}+\frac{1}{s}K_{2R} \end{bmatrix}
\end{aligned}\right\} . \qquad (64)$$

Since

$$\overset{\#}{z} = \begin{bmatrix} z_{11} & 0 \\ 0 & z_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{\omega}} Z_{11} & 0 \\ 0 & \sqrt{\omega}\, Z_{22} \end{bmatrix} \qquad (65)$$

then $z_{k(11)} = \frac{1}{\sqrt{\omega}} Z_{k(11)}$, $z_{k(22)} = \sqrt{\omega}\, Z_{k(22)}$

where $k = 0, 1, 2$ and $s = i\omega$.

By substituting Eqs. (64) and (65) into the quasi-matching condition as expressed by Eq. (62), the "best" values for the support parameters for the shaft system shown in Figure 19 can be obtained. Thus,

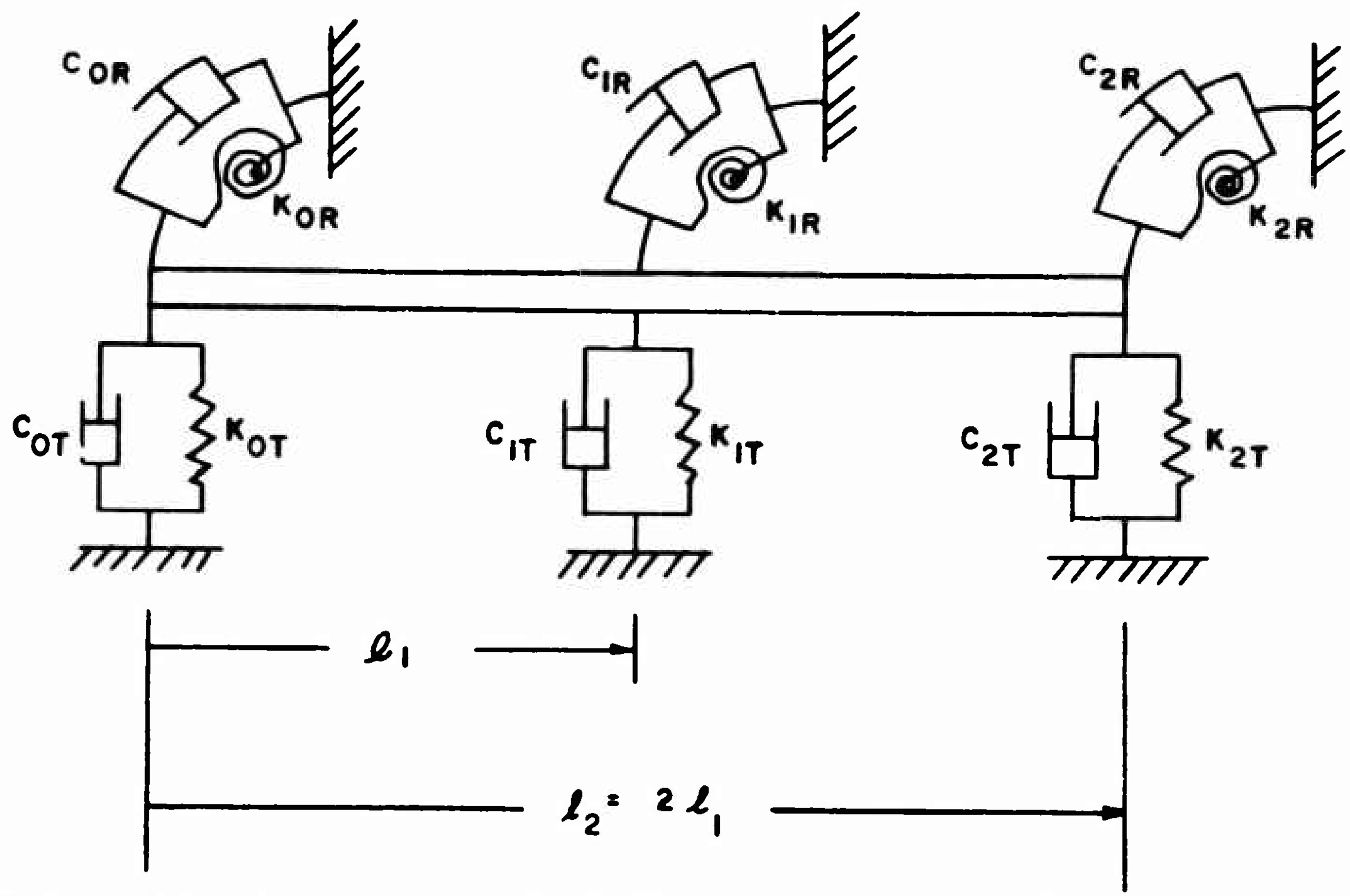

Figure 19. Three - support Shaft System With Specified Support Configurations.

$$\frac{1}{\sqrt{\omega}}\left[ie_1e_2(1+e_3^2)-(e_1e_3^2+ie_2)Q_{22}\right](C_{1T}-\frac{i}{\omega}K_{1T})+$$

$$\sqrt{\omega}\left[(1+e_3^2)-(e_1e_3^2+ie_2)Q_{11}\right](C_{1R}-\frac{i}{\omega}K_{1R})-$$

$$(e_1e_3^2+ie_2)\left[(C_{1T}-\frac{i}{\omega}K_{1T})(C_{1R}-\frac{i}{\omega}K_{1R})\right]+$$

$$\left[(1+e_3^2)(Q_{22}+ie_1e_2Q_{11})-2e_3(e_2+ie_1)Q_{12}+(e_1e_3^2+ie_2)(Q_{12}^2-Q_{11}Q_{22})-\right.$$

$$\left.(e_1+ie_2e_3^2)\right]=0 \tag{66}$$

where

$$Q_{11}=\frac{(1+e_3^2)}{\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]}\left[\Gamma_{2(11)}(\ell_2)e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}-1\right],$$

$$Q_{22}=\frac{ie_1e_2(1+e_3^2)}{\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]}\left[\Gamma_{2(11)}(\ell_2)e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}+1\right],$$

$$Q_{12}=\frac{e_3}{\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]}\left[-(e_2+ie_1)\Gamma_{2(11)}(\ell_2)e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}+(e_2-ie_1)\right],$$

$$\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]=-e_1e_3^2\left[e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_{2(11)}(\ell_2)+1\right]-$$
$$ie_2\left[e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_{2(11)}(\ell_2)-1\right],$$

$$\overset{\#}{\Gamma}_{2(11)}(\ell_2)=\frac{1}{\det\left[-\overset{\#}{C}_{22}+\overset{\#}{Z}_2(\ell_2)\overset{\#}{C}_{12}\right]}\left\{(1+e_3^2)\left[\sqrt{\omega}(C_{2R}-\frac{i}{\omega}K_{2R})+\right.\right.$$

$$\left.\frac{ie_1e_2}{\sqrt{\omega}}(C_{2T}-\frac{i}{\omega}K_{2T})\right]-(e_1e_3^2+ie_2)(C_{2T}-\frac{i}{\omega}K_{2T})(C_{2R}-\frac{i}{\omega}K_{2R})-$$

$$\left.(e_1+ie_2e_3^2)\right\},$$

$$\det\left[-\overset{\#}{C}_{22}+\overset{\#}{Z}_2(\ell_2)\overset{\#}{C}_{12}\right] = -ie_3^2\left[1+\frac{e_1}{\sqrt{\omega}}(C_{2T}-\frac{i}{\omega}K_{2T})\right]\left[e_2+i\sqrt{\omega}(C_{2R}-\frac{i}{\omega}K_{2R})\right] +$$

$$\left[1-\frac{ie_2}{\sqrt{\omega}}(C_{2T}-\frac{i}{\omega}K_{2T})\right]\left[e_1+\sqrt{\omega}(C_{2R}-\frac{i}{\omega}K_{2R})\right] .$$

As can be seen from the above expressions, the only unknowns involved are K_{1T}, K_{1R}, C_{1T}, C_{1R} of the interior support, provided that both end support configurations are given. Hence, Eq. (66) provides a means of establishing the values that should be used for the unknowns K_{1T}, K_{1R}, C_{1T}, and C_{1R} in order to satisfy the quasi-matching condition for the interior support in the shaft system shown in Figure 19. The interesting point here is that the set of values for K_{1T}, K_{1R}, C_{1T}, C_{1R} required to satisfy Eq. (66) is not unique. This permits a wide range of selection of these parameters to match the feasibility and availability of various damper and spring materials and configurations.

For a shaft system similar to the one shown in Figure 19, but which has no rotational spring and damper units attached to its supports, the quasi-matching condition of the interior support can be written simply by setting $K_{kR} = C_{kR} = 0$ and $k = 0, 1, 2$, in Eq. (66):

$$C_{1T} - \frac{i}{\omega}K_{1T} + Q = 0 \tag{67}$$

where

$$Q = \frac{\sqrt{\omega}}{ie_1e_2(1+e_3^2)-(e_1e_3^2+ie_2)Q_{22}}\Big[(1+e_3^2)(Q_{22}+ie_1e_2Q_{11})-2e_3(e_2+ie_1)Q_{12} +$$

$$(e_1e_3^2+ie_2)(Q_{12}^2-Q_{11}Q_{22})-(e_1+ie_2e_3^2)\Big] ,$$

Q_{11}, Q_{22}, Q_{12}, $\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]$ have the same expressions as in Eq. (66),

$$\Gamma_{2(11)}(\ell_2) = \frac{1}{\det\left[-\overset{\#}{C}_{22}+\overset{\#}{Z}_2(\ell_2)\overset{\#}{C}_{12}\right]}\left[\frac{ie_1e_2(1+e_3^2)}{\sqrt{\omega}}(C_{2T}-\frac{i}{\omega}K_{2T})-(e_1+ie_2e_3^2)\right] ,$$

$$\det\left[-\overset{\#}{C}_{22}+\overset{\#}{Z}_2(\ell_2)\overset{\#}{C}_{12}\right] = \frac{-ie_1e_2(1+e_3^2)}{\sqrt{\omega}}(C_{2T}-\frac{i}{\omega}K_{2T})+(e_1-ie_2e_3^2) .$$

The quantity Q is a given constant which relates to end configurations, shaft characteristics, and rotating speed. Eq. (67) demonstrates the usefulness of the transmission line analogy by quasi-matching; C_{1T} and K_{1T} are the only unknowns appearing in a linear equation and can be determined directly.

In the cases for which the interior support is placed in close proximity of the right end support, Eqs. (66) and (67) can be used also for establishing the optimum impedance terms of the interior support by letting $\Gamma_{2(11)}(\ell_2) = 0$.

<u>Simplification of Quasi-Matching Conditions by Neglecting Rotary Inertia, Shear, and Gyroscopic Effects</u>

It can be verified that the differential equation accounting for the bending effects in the shaft only can be obtained from Eq. (7) simply by setting $e_1 = e_2 = e_3 = 1$. In practice, if the rotating speed is not extremely high, the terms for rotational inertia, shear deformation, and gyroscopic effects can be neglected. When the above argument is used, the quasi-matching equations for a shaft with one interior support may be simplified accordingly. With reference to Eq. (62) and Figure 16,

$$2(z_{1r(22)}+iz_{1r(11)})+(1+i)(z_{1r(12)}z_{1r(21)}-z_{1r(11)}z_{1r(22)}-z_{1r(12)}-z_{1r(21)}-1)=0 \tag{68}$$

where

$$z_{1r(11)} = z_{1(11)}+\frac{2}{\Delta_1}\left[\Gamma_{2(11)}(\ell_2)e^{-2i\sqrt{\omega}(\ell_2-\ell_1)}-1\right]$$

$$z_{1r(12)} = z_{1r(21)} = \frac{1}{\Delta_1}\left[-(1+i)\Gamma_{2(11)}(\ell_2)e^{-2i\sqrt{\omega}(\ell_2-\ell_1)}+(1-i)\right]$$

$$z_{1r(22)} = z_{1(22)} + \frac{2i}{\Delta_1}\left[\Gamma_{2(11)}(\ell_2)e^{-2i\sqrt{\omega}(\ell_2-\ell_1)} + 1\right]$$

$$\Delta_1 = -\left[e^{-2i\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_{2(11)}(\ell_2)+1\right] - i\left[e^{-2i\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_{2(11)}(\ell_2) - 1\right]$$

$$\Gamma_{2(11)}(\ell_2) = \frac{1}{\Delta_2}\Big[2(z_{2(22)}+iz_{2(11)})+(1+i)(z_{2(12)}z_{2(21)}-z_{2(11)}z_{2(22)}-$$
$$z_{2(12)}-z_{2(21)}-1)\Big]$$

$$\Delta_2 = -\left[i + iz_{2(11)} + z_{2(12)}\right]\left[1 - iz_{2(21)} + iz_{2(22)}\right] +$$
$$\left[1 + iz_{2(21)} + z_{2(22)}\right]\left[1 - iz_{2(11)} + iz_{2(12)}\right] .$$

For the specific shaft configuration of Figure 19, the corresponding quasi-matching condition can be rewritten in the following simplified form (see Eq. (66)):

$$\frac{1}{\sqrt{\omega}}\left[2i-(1+i)Q_{22}\right](C_{1T}-\frac{i}{\omega}K_{1T}) + \sqrt{\omega}\left[2-(1+i)Q_{11}\right](C_{1R}-\frac{i}{\omega}K_{1R}) -$$
$$(1+i)(C_{1T}-\frac{i}{\omega}K_{1T})(C_{1R}-\frac{i}{\omega}K_{1R}) + \left[2(Q_{22}+iQ_{11}) +\right.$$
$$\left.(1+i)(Q_{12}^2-Q_{11}Q_{22}-2Q_{12}-1)\right] = 0 \tag{69}$$

where

$$Q_{11} = \frac{2}{\Delta_1}\left[\Gamma_{2(11)}(\ell_2)e^{-2i\sqrt{\omega}(\ell_2-\ell_1)} - 1\right]$$

$$Q_{22} = \frac{2i}{\Delta_1}\left[\Gamma_{2(11)}(\ell_2)e^{-2i\sqrt{\omega}(\ell_2-\ell_1)} + 1\right]$$

$$Q_{12} = \frac{1}{\Delta_1}\left[-(1+i)\Gamma_{2(11)}(\ell_2)e^{-2i\sqrt{\omega}(\ell_2-\ell_1)} + (1-i)\right]$$

$$\Delta_1 = -\left[e^{-2i\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_{2(11)}(\ell_2) + 1\right] - i\left[e^{-2i\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_{2(11)}(\ell_2) - 1\right]$$

$$\Gamma_{2(11)}(\ell_2) = \frac{1}{\Delta_2}\left\{2\left[\sqrt{\omega}(C_{2R}-\frac{i}{\omega}K_{2R}) + \frac{i}{\sqrt{\omega}}(C_{2T}-\frac{i}{\omega}K_{2T})\right] -\right.$$
$$\left.(1+i)\left[(C_{2T}-\frac{i}{\omega}K_{2T})(C_{2R}-\frac{i}{\omega}K_{2R}) + 1\right]\right\}$$

$$\Delta_2 = -i\left[1 + \frac{1}{\sqrt{\omega}}(C_{2T}-\frac{i}{\omega}K_{2T})\right]\left[1 + i\sqrt{\omega}(C_{2R}-\frac{i}{\omega}K_{2R})\right] +$$
$$\left[1 - \frac{i}{\sqrt{\omega}}(C_{2T}-\frac{i}{\omega}K_{2T})\right]\left[1 + \sqrt{\omega}(C_{2R}-\frac{i}{\omega}K_{2R})\right] .$$

For the simplified shaft configuration in which no rotational springs and dampers are being used, a very simple quasi-matching equation for determining optimum damping and spring coefficients in the interior support can be obtained. With reference to Eq. (67),

$$C_{1T} - \frac{i}{\omega} K_{1T} + Q = 0 \tag{70}$$

where

$$Q = \frac{\sqrt{\omega}}{2i-(1+i)Q_{22}} \left[2(Q_{22}+iQ_{11})+(1+i)(Q_{12}^2 - Q_{11}Q_{22} - 2Q_{12} - 1) \right]$$

Q_{11}, Q_{22}, Q_{12} have the same expressions as in Eq. (69),

$$\Gamma_{2(11)}(\ell_2) = \frac{1}{\Delta_2} \left[\frac{2i}{\sqrt{\omega}} (C_{2T} - \frac{i}{\omega} K_{2T}) - (1+i) \right]$$

$$\Delta_2 = - \frac{2i}{\sqrt{\omega}} (C_{2T} - \frac{i}{\omega} K_{2T}) + (1-i) \quad .$$

Thus, fairly straightforward solutions have been obtained which permit the designer to obtain in a direct manner the spring and damping coefficients required in the interior support for the minimum vibration response operation of a hypercritical shaft on three flexible supports. To suit cases other than those discussed specifically in this report, Eq. (68) can be manipulated. Again, it should be noted that for the cases in which the interior support is placed close to the right end support, Eqs. (68), (69), and (70) can be simplified further by letting $\Gamma_{2(11)}(\ell_2) = 0$. It should be pointed out also that the effect of mass in the interior support can be included directly in Eqs. (66), (67), (69), and (70) by replacing the spring coefficients:

$$K_{1T} = (K_{1T} - \omega^2 M_1) \quad ,$$

$$K_{2T} = (K_{2T} - \omega^2 M_2) \quad ,$$

$$K_{1R} = (K_{1R} - \omega^2 I_1) \quad ,$$

$$K_{2R} = (K_{2R} - \omega^2 I_2) \quad ,$$

where M_1 and M_2 are, respectively, the mass of the support systems at $x = \ell_1$ and $x = \ell_2$, and I_1 and I_2 are, respectively, the mass moments of the support systems about their center of gravity at $x = \ell_1$ and $x = \ell_2$ of the shaft.

Design Formula for the Shaft with Its Interior Support Placed in Close Proximity of One of Its End Supports

For the case in which the shaft is being treated essentially as a two-support system (i. e., the interior support is placed in close proximity of one of the end supports), it is possible to develop a very convenient formula for the determination of the optimum quantities (in terms of quasi-matching) which the interior support must provide for a minimum vibration response of the shaft. In this case, it is assumed that $e_1 = e_2 = e_3 = 1$. If the impedance of the interior support is considered to act in a translational fashion only, the impedance matrix of the interior support can be written as follows:

$$\overset{\#}{Z}_1(\ell_1) = \begin{bmatrix} C_{1T} - \frac{i}{\omega}(K_{1T} - \omega^2 M_1) & 0 \\ 0 & 0 \end{bmatrix} .$$

Substitution of this condition in Eq. (70), and letting $\Gamma_{2(11)}(\ell_2) = 0$, results, after some algebraic work, in the following formula:

$$C_{1T} = \frac{1}{2\omega}\left\{\omega^3 + \left[\omega^{3/2} + 2(M_1\omega^2 - K_{1T})\right]^2\right\}^{1/2} \qquad (71)$$

where

$$C_{1T} = \frac{C^*_{1T} c_s}{E_y A}$$

$$\omega = \frac{\omega^* R_b}{c_s}$$

$$M_1 = \frac{M^*_1}{\rho A R_b}$$

$$K_{1T} = \frac{K^*_{1T} R_b}{E_y A}$$

$$c_s = \sqrt{\frac{E_y}{\rho}} .$$

Also, C^*_{1T}, M^*_1, and K^*_{1T} are, respectively, the dimensional quantities for the damping coefficient, mass, and spring coefficient of the interior

support, and ω^* is the dimensional quantity for the angular velocity of the shaft.

Eq. (71) will calculate the optimum damping coefficients for preselected sets of masses, and spring coefficients in the support in terms of the angular velocity of the shaft. It should be noted that the values for C_{1T} are independent of the impedance conditions of the end supports and that the values for optimum damping for two interior supports placed close to each end support are the same as that for one interior support placed close to one or the other end support.

support, and ω^* is the dimensional quantity for the angular velocity of the shaft.

Eq. (71) will calculate the optimum damping coefficients for preselected sets of masses, and spring coefficients in the sunport in terms of the angular velocity of the shaft. It should be noted that the values for C_{1T} are independent of the impedance conditions of the end supports and that the values for optimum damping for two interior supports placed close to each end support are the same as that for one interior support placed close to one or the other end support.

CHAPTER 5

CONCLUSIONS

EXTENSION OF TRANSMISSION LINE ANALOGY

The transmission line analogy can be extended to shafts having any number of interior supports. However, the solution is considerably more complicated for shafts having more than one interior support.

MATCHED END IMPEDANCES

For the shaft with both end impedances matched with the characteristic impedance of the shaft. no intermediate support is needed to assist in the minimization of vibration response.

QUASI-MATCHING

If it is physically impossible or impractical to terminate a shaft in its characteristic impedance, quasi-matched end impedances or quasi-matched interior supports (when end conditions are not available for optimization) should provide good performance. Quasi-matching involves the selection of support conditions in such a way that the predominant Γ_{11} term of the reflection matrix vanishes. The concept of quasi-matching is based on the assumption that $\exp(-e_2\sqrt{\omega}\,(\ell_1-\ell_{i-1})) \ll 1$, which is true as ω increases.

ONE OPTIMUM INTERIOR SUPPORT

When the shaft and end support impedances are not matched and when only one interior support is used, two approaches may be employed to assist in the minimization of vibration response:

1. If both ends have the same configuration, the interior support may be placed at the midspan, provided the conditions of this interior support obey

$$\Gamma_{1r(11)}(\ell_1) \equiv \Gamma_{1\ell(11)}(\ell_1) = 0$$

as indicated by Eq. (62).

2. If one end support is different from the other, the closer the matched intermediate support is placed to one of the ends, the more effectively it will minimize the vibration response of the shaft with provision that:

   a. If the interior support is placed close to the left end, it should be arranged such that $\Gamma_{1\ell(11)}(\ell_1) = 0$.

   b. If the interior support is placed close to the right end, it should be arranged such that $\Gamma_{1r(11)}(\ell_1) = 0$. The equation

for $\Gamma_{1r(11)}(\ell_1) = 0$ will have the same form as Eq. (62), except that ℓ_1 (see Figure 4) indicates the actual location of the interior support.

TWO OPTIMUM INTERIOR SUPPORTS

When the shaft and end support impedances are not matched, the use of two matched intermediate supports placed close to the ends of the shaft is recommended. This is equivalent to letting $n = 3$ for the shaft system shown by Figure 4, in which ℓ_1, ℓ_2 should approach 0, ℓ_3, respectively, and the 1^{st} and the 2^{nd} interior supports should be designed in terms of configurations which satisfy the conditions

$$\Gamma_{1\ell(11)}(\ell_1) = 0$$

and

$$\Gamma_{2r(11)}(\ell_2) = 0 \quad .$$

In other words, only the incident wave exists on $[\ell_1 , \ell_2]$, which is almost equal in magnitude to $[0 , \ell_3]$, namely, the whole portion of the shaft.

BIBLIOGRAPHY

1. Abramson, H. N., "The Propagation of Flexural Elastic Waves in Solid Circular Cylinders," Journal Acoustical Society of America, Volume 29, January 1957, pp. 42-46.

2. Battelle Memorial Institute, The, Design Manual for Supercritical-Speed Power-Transmission Shafts, prepared for the U.S. Army Transportation Research Command and U.S. Air Force Research and Technology Division, January 1965.

3. Bewley, L. V., Traveling Waves on Transmission System, John Wiley and Sons, Inc., Englewood, New Jersey, 1951, pp. 40-105.

4. Dowell, J. C., "Attenuation and Successive Reflections of Traveling Waves," A.I.E.E. Transactions, Volume 50, pp. 551.

5. Dubensky, R. G., Mellor, C. C., Jr., and Voorhees, J. E., Design Criteria for High-Speed Power-Transmission Shafts, Technical Documentary Report No. ASD-TDR-62-728, The Battelle Memorial Institute, Columbus, Ohio, Part I, January 1963, Part II, December 1964.

6. Eppink, R. T., "An Investigation of the Behavior of Floating Ring Dampers and the Dynamics of Hypercritical Shafts on Flexible Supports," First Quarterly Report, Report No. CE-3340-101-64U (Revised), Research Laboratories for the Engineering Sciences, University of Virginia, Charlottesville, Virginia, February 1964.

7. Eppink, R. T., and Friedericy, J. A., "An Investigation of the Behavior of Floating Ring Dampers and the Dynamics of Hypercritical Shafts on Flexible Supports," Second Quarterly Report, Report No. CE-3340-102-64U, Research Laboratories for the Engineering Sciences, University of Virginia, Charlottesville, Virginia, March 1964 and Third Quarterly Report, Report No. CE-3340-103-64U, May 1964.

8. Friedericy, J. A., Eppink, R. T., Liu, Y. N., and Cetiner, A., An Investigation of the Behavior of Floating Ring Dampers and the Dynamics of Hypercritical Shafts on Flexible Supports, USAAML Technical Report 65-34, U.S. Army Aviation Materiel Laboratories, Fort Eustis, Virginia, June 1965.

9. Jones, R. P. N., "The Wave Method for Solving Flexural Vibration Problems," Journal of Applied Mechanics, Volume 21, Number 1, March 1954, pp. 75-80.

10. Leonard, R. W., and Budiansky, B., "On Traveling Waves in Beams," NACA Technical Note 2874, National Aeronautics and Space Administration, Hampton, Virginia, 1953.

11. Lowry, R. A., and Parker, H. M., "Performance of Long Shafts Operating at Critical Speeds," Report No. ORL-1779-59TR1, Research Laboratories for the Engineering Sciences, University of Virginia, Charlottesville, Virginia, April 1959.

12. Miklowitz, J., "Flexural Wave Solutions of Coupled Equations Representing the More Exact Theory of Bending," Journal of Applied Mechanics, Volume 20, Number 4, December 1953, pp. 511-514.

13. Moore, R. K., Traveling-Wave Engineering, McGraw-Hill Book Company, New York, New York, 1960.

14. Nelson, L. W., "Flexural Vibration of Uniform Shafts," Report No. EP-4422-184-61S (Classified), Research Laboratories for the Engineering Sciences, University of Virginia, Charlottesville, Virginia, November 1961. (Declassified July 1962.)

15. Eppink, R. T., "Mathematical Details, Notes, and Comments," on report in preceding reference, Research Laboratories for the Engineering Sciences, University of Virginia, Charlottesville, Virginia, July 1962.

16. Potter, J. I., and Fich, S., Theory of Networks and Lines, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963, pp. 255-374.

17. Ripperger, E. A., and Abramson, H. N., "A Study of the Propagation of Flexural Waves in Elastic Beams," Journal of Applied Mechanics, Volume 24, Number 3, September 1957, pp. 431-434.

18. Snowdon, J. C., "Mechanical Impedance and Transmissibility of Simply Supported Beams," Journal Acoustical Society of America, Volume 35, Number 2, February 1963, pp. 228-233.

19. Snowdon, J. C., "Transverse Vibration of Beams with Internal Damping, Rotary Inertia, and Shear," Journal Acoustical Society of America, Volume 35, Number 12, December 1963, pp. 1997-2006.

20. Snowdon, J. C., "Approximate Expressions for the Mechanical Impedance and Transmissibility of Beam Vibrating in Their Transverse Modes," Journal Acoustical Society of America, Volume 36, Number 2, February 1964, pp. 366-375.

21. Voorhees, J. E., Mellor, C. C., Jr., and Dubensky, R. G., "The Control of Shaft Vibrations at Their Hypercritical Speeds," paper presented at the Design Engineering Conference and Show of the American Society of Mechanical Engineers, New York, May 20-23, 1963.

APPENDIX A

MATHEMATICAL DERIVATIONS OF THE ROTATORY AND GYROSCOPIC EFFECTS

In Figures 20, 21 and 22, a right-hand screw convention is used to indicate the positive directions of the applying torques on the differential mass element of a rotating shaft. If it is assumed that the transverse deflections (Y^*_{11} and Y^*_{12}) of the shaft during vibration are very small and that the center of gravity, o, of the differential mass element coincides with the axis of the shaft, the position of this element will be completely determined by the coordinates Y^*_{11} and Y^*_{12} of its center of gravity, o, and by the angular rotations, Y^*_{21} and Y^*_{22} .

The conditions assumed here correspond to the case of a vertical shaft when the weight of the differential mass element does not affect the deflections of the shaft. Under these conditions, if W equals the weight of the differential mass element and if only the elastic reaction of the shaft is taken into consideration, the equations of motion of the center of gravity of the differential mass element are as follows (see Figures 21 and 22):

In $-X_3$ direction,

$$\text{Inertia force} = \frac{W}{g} Y^*_{12tt}$$

In $-X_2$ direction,

$$\text{Inertia force} = \frac{W}{g} Y^*_{11tt}$$

where $\frac{W}{g} = \rho A dx$.

The equations of relative motion of the differential mass element with respect to its center of gravity, o, will now be obtained by using the principle of angular momentum, which states that the rate of increase of the total moment of momentum of any moving system about any fixed axis is equal to the total moment of the external forces about this axis. In calculating the rate of change of the angular momentum about a fixed axis drawn through the instantaneous position of the center of gravity, o, the relative motion alone can be taken into consideration.

In calculating the components of the angular momentum, the principal axes of inertia of the differential mass element will be taken. The axis of rotation, oo, is one of these axes. The two other axes, oa and ob, will be any two perpendicular diameters of the element.

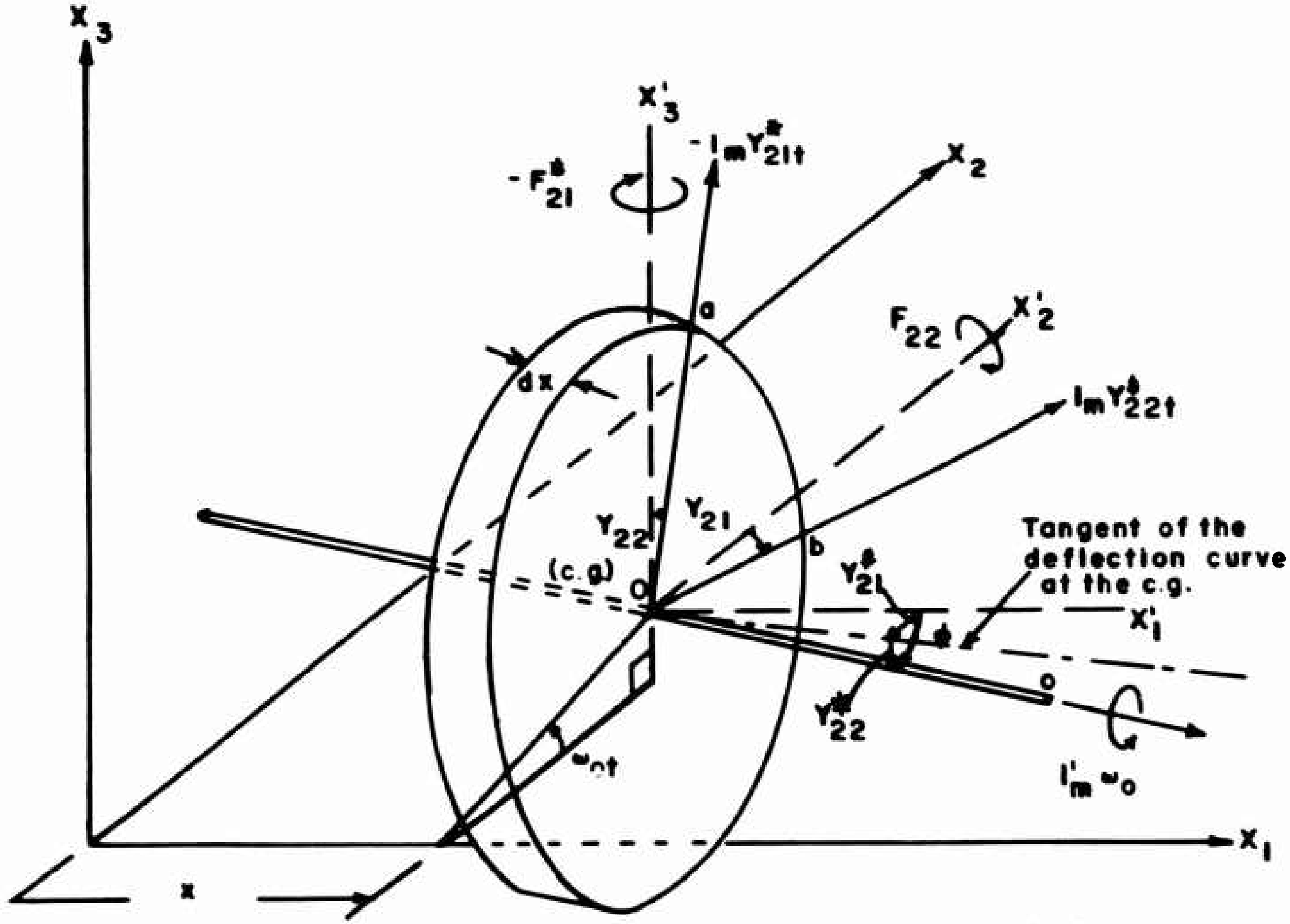

Figure 20. Moments Induced by Angular Movements Acting on a Differential Mass Element.

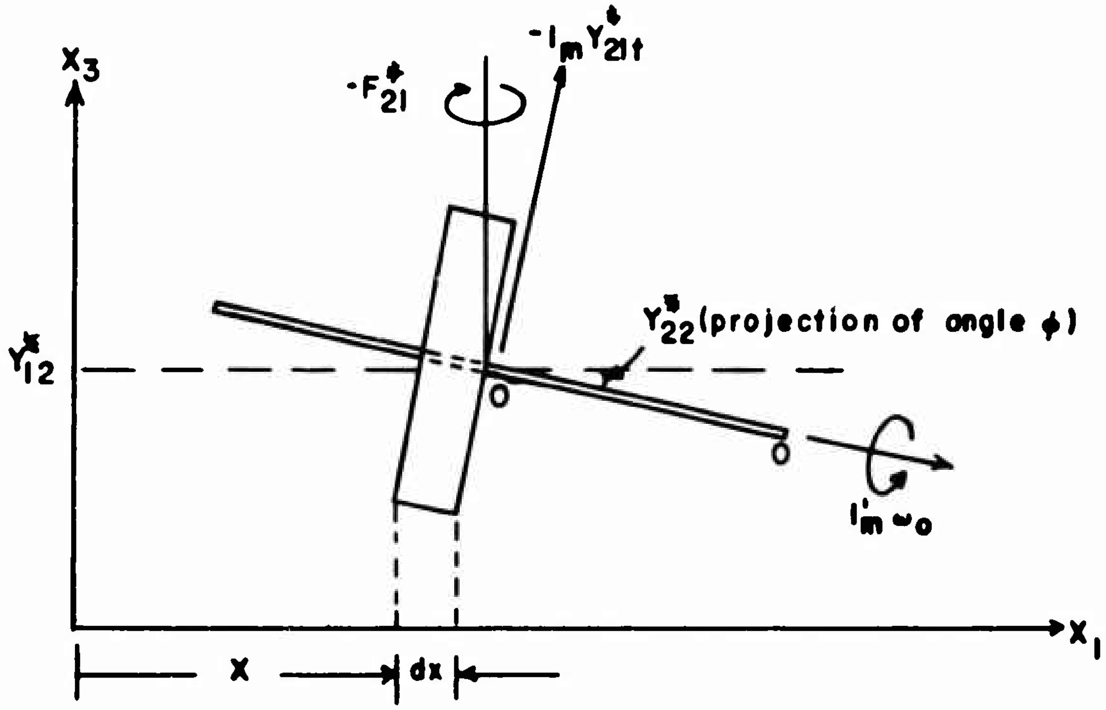

Figure 21. Projection on Plane X_1X_2.

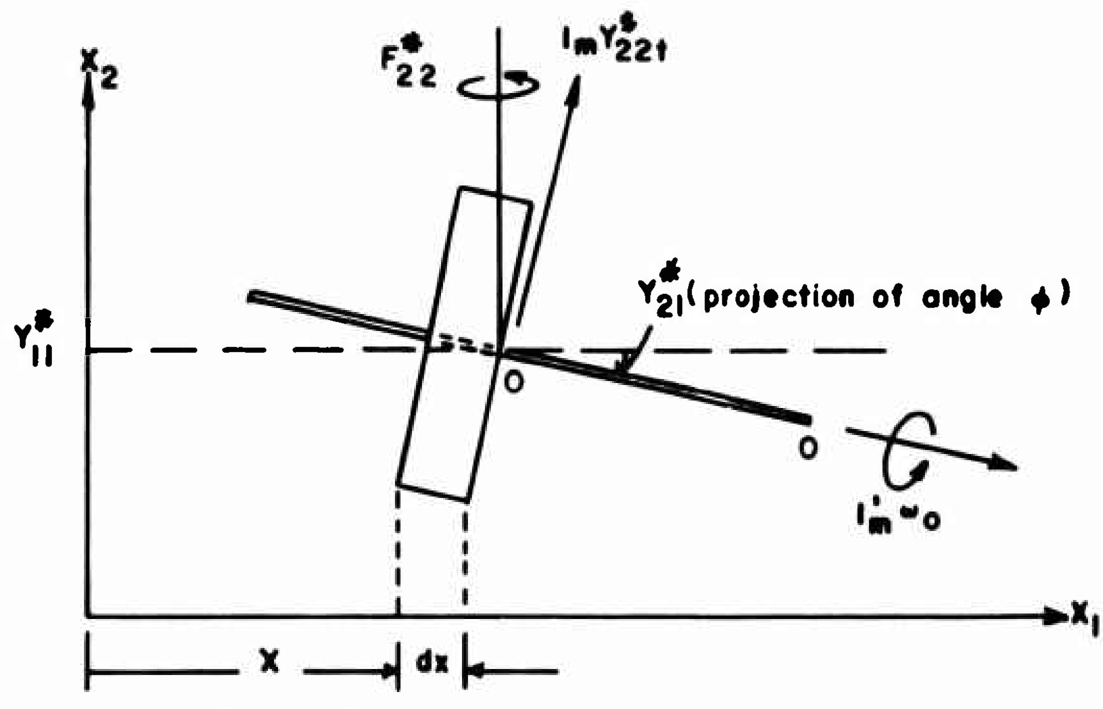

Figure 22. Projection on Plane X_1X_2.

Let

I'_m = mass moment of inertia of the mass element about the axis oo,

I_m = mass moment of inertia of the mass element about a diameter.

Observe that

$I'_m = 2I_m$

$I'_m\omega_o$ = component of angular momentum about the axis oo,

$-I_m Y^*_{21t}$ = component of angular momentum about the axis oa,

$I_m Y^*_{22t}$ = component of angular momentum about the axis ob.

Since Y^*_{21} and Y^*_{22} are assumed to be small, then Y^*_{21t} and Y^*_{22t} will be approximate values of the angular velocities about the diameters oa and ob. If components of the angular momentum on the fixed axes oX'_2 and oX'_3 are projected through the instantaneous position of the center of gravity, o, the resultants of the angular momentum in the X_3- and X_2-directions can be obtained. It is shown that $\cos Y^*_{22}$ and $\cos Y^*_{21} \approx 1$ and that $\sin Y^*_{22} \approx Y^*_{22}$ and $\sin Y^*_{21} \approx Y^*_{21}$.

In the X_3-direction, with reference to X_1X_3-plane projection (see Figure 21),

$$(-I_m Y^*_{21t})(\cos Y^*_{22})-(I'_m\omega_o)(\sin Y^*_{22}) \approx -I_m Y^*_{21t}-I'_m\omega_o Y^*_{22} \quad .$$

In the X_2-direction, with reference to X_1X_2-plane projection (see Figure 22),

$$(I_m Y^*_{22t})(\cos Y^*_{21})-(I'_m\omega_o)(\sin Y^*_{21}) \approx I_m Y^*_{22t}-I'_m\omega_o Y^*_{21} \quad .$$

Then, from the principle of angular momentum and since $I'_m = 1I_m$, the following can be shown:

In X_3-direction,

$$-F^*_{21} = \frac{d}{dt}(-I_m Y^*_{21t}-I'_m\omega_o Y^*_{22})$$

or

$$F^*_{21} = I_m Y^*_{21tt} + 2I_m \omega_o Y^*_{22t} \quad .$$

In X_2-direction,

$$F^*_{22} = \frac{d}{dt}(I_m Y^*_{22t} - I'_m \omega_o Y^*_{21})$$

or

$$F^*_{22} = I_m Y^*_{22tt} - 2I_m \omega_o Y^*_{21t} \quad .$$

If I is the moment of inertia of the cross section of the differential mass element,

$$I_m = \rho I dx$$

or

$$I'_m = 2I_m = 2\rho I dx \quad .$$

Then, the above expressions for moment of momentum may be rewritten as:

In X_3-direction,

$$F^*_{21} = \rho I Y^*_{21tt} dx + 2I\omega_o \rho Y^*_{22t} dx$$

In X_2-direction,

$$F^*_{22} = \rho I Y^*_{22tt} dx - 2I\omega_o \rho Y^*_{21t} dx$$

which were shown in Figures 2 and 3 in Chapter 1.

$$F^*_{21} = I_m Y^*_{21tt} + 2I_m \omega_o Y^*_{22t} \quad .$$

In X_2-direction,

$$F^*_{22} = \frac{d}{dt}(I_m Y^*_{22t} - I'_m \omega_o Y^*_{21})$$

or

$$F^*_{22} = I_m Y^*_{22tt} - 2I_m \omega_o Y^*_{21t} \quad .$$

If I is the moment of inertia of the cross section of the differential mass element,

$$I_m = \rho I dx$$

or

$$I'_m = 2I_m = 2\rho I dx \quad .$$

Then, the above expressions for moment of momentum may be rewritten as:

In X_3-direction,

$$F^*_{21} = \rho I Y^*_{21tt} dx + 2I\omega_o \rho Y^*_{22t} dx$$

In X_2-direction,

$$F^*_{22} = \rho I Y^*_{22tt} dx - 2I\omega_o \rho Y^*_{21t} dx$$

which were shown in Figures 2 and 3 in Chapter 1.

APPENDIX B

IMPORTANT FUNCTIONS

$$e_3 = \sqrt{1+\left[\omega_o + \left(\frac{e'-1}{2}\right)\right]^2} + \left[\omega_o + \left(\frac{e'-1}{2}\right)\right]$$

$$e_1 = \sqrt{\frac{1}{e_3} + e'\omega}$$

$$e_2 = \frac{\sqrt{1+e'\omega(2\omega_o - \omega)}}{e_1}$$

where $\omega = -is$ and $\omega_o = \omega$ for steady-state condition.

$$\overset{\#}{C}_{11} = \begin{bmatrix} -ie_1e_3 & -ie_2 \\ 1 & -ie_3 \end{bmatrix} \qquad \overset{\#}{C}_{22} = \begin{bmatrix} ie_3 & -1 \\ e_1 & -e_2e_3 \end{bmatrix}$$

$$\overset{\#}{C}_{12} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \overset{\#}{C}_{11} \qquad \overset{\#}{C}_{21} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \overset{\#}{C}_{22}$$

$$\overset{\#}{C}_{+} = \frac{\begin{bmatrix} ie_3 & 1/e_1 \\ 1 & -e_3/e_2 \end{bmatrix}}{2(1+e_3^2)} \qquad \overset{\#}{C}_{y} = \omega^p \begin{bmatrix} \omega & 0 \\ 0 & \omega^{3/2} \end{bmatrix}$$

$$\overset{\#}{C}_{-} = \overset{\#}{C}_{+} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \overset{\#}{C}_{f} = \omega^p \begin{bmatrix} \omega^{3/2} & 0 \\ 0 & \omega \end{bmatrix}$$

where p is an arbitrary real number which, once chosen, is the same for each p.

$$\overset{\#}{z}_s = \frac{1}{\sqrt{\omega}} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\omega} \end{bmatrix} \overset{\#}{Z}_s \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\omega} \end{bmatrix}$$

$\overset{\#}{Z}_s$ = characteristic impedance of the rotating shaft

$$\overset{\#}{\mathcal{Z}}_s = \overset{\#}{C}_{21}\overset{\#}{C}{}_{11}^{-1}$$

$$\overset{\#}{\mathcal{Z}}_s = \left(\frac{1+e_3^2}{e_2^2+e_1^2e_3^4}\right)\begin{bmatrix} (e_1e_3^2+ie_2) & -e_3\left(e_1e_2+i\dfrac{e_2^2-e_1^2e_3^2}{1+e_3^2}\right) \\ -e_3\left(e_1e_2+i\dfrac{e_2^2-e_1^2e_3^2}{1+e_3^2}\right) & e_1e_2(e_2-ie_1e_3^2) \end{bmatrix}.$$

APPENDIX C

MATHEMATICAL DERIVATIONS OF APPLYING BOUNDARY CONDITIONS AT SUPPORTS

GENERAL SOLUTION

The general solution of the equations of motion is shown in Eq. (10) as

$$\begin{bmatrix} \tilde{Y} \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x) & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{R}(-x) \end{bmatrix} \begin{bmatrix} \bar{q} \\ \bar{r} \end{bmatrix}$$

and is written out as

$$\left.\begin{aligned} \tilde{Y}(x) &= \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{R}(x)\bar{q}+\overset{\#}{C}_{12}\overset{\#}{R}(-x)\bar{r}\right] \\ \tilde{F}(x) &= \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{R}(x)\bar{q}+\overset{\#}{C}_{22}\overset{\#}{R}(-x)\bar{r}\right] \end{aligned}\right\}$$

If the general solution is applied to each span, the following set of equations can be obtained with integration constants $\bar{q}_1$, $\bar{q}_2$, . . . , $\bar{q}_{j-1}$, $\bar{q}_j$, $\bar{q}_{jr}$, $\bar{q}_{j+1}$, . . . , $\bar{q}_{n-1}$, $\bar{q}_n$, $\bar{r}_1$, $\bar{r}_2$, . . . , $\bar{r}_{j-1}$, $\bar{r}_{j\ell}$, $\bar{r}_{jr}$, $\bar{r}_{j+1}$, . . . , $\bar{r}_{n-1}$, $\bar{r}_n$ (see Figure 4):

For $[0, \ell_1]$,

$$\tilde{Y}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{R}(x)\bar{q}_1+\overset{\#}{C}_{12}\overset{\#}{R}(-x)\bar{r}_1\right] ,$$

$$\tilde{F}(x) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{R}(x)\bar{q}_1+\overset{\#}{C}_{22}\overset{\#}{R}(-x)\bar{r}_1\right] ,$$

For $[\ell_{j-2}, \ell_{j-1}]$,

$$\tilde{Y}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{R}(x)\bar{q}_{j-1}+\overset{\#}{C}_{12}\overset{\#}{R}(-x)\bar{r}_{j-1}\right] ,$$

$$\tilde{F}(x) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{R}(x)\bar{q}_{j-1}+\overset{\#}{C}_{22}\overset{\#}{R}(-x)\bar{r}_{j-1}\right] ,$$

For $[\ell_{j-1}, a_{jk}]$,

$$\overset{\simeq}{Y}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{R}(x)\bar{q}_{j\ell}+\overset{\#}{C}_{12}\overset{\#}{R}(-x)\bar{r}_{j\ell}\right] ,$$

$$\overset{\simeq}{F}(x) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{R}(x)\bar{q}_{j\ell}+\overset{\#}{C}_{22}\overset{\#}{R}(-x)\bar{r}_{j\ell}\right] ,$$

For $[a_{jk} , \ell_j]$,

$$\overset{\simeq}{Y}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{R}(x)\bar{q}_{jr}+\overset{\#}{C}_{12}\overset{\#}{R}(-x)\bar{r}_{jr}\right] ,$$

$$\overset{\simeq}{F}(x) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{R}(x)\bar{q}_{jr}+\overset{\#}{C}_{22}\overset{\#}{R}(-x)\bar{r}_{jr}\right] ,$$

For $[\ell_j , \ell_{j+1}]$,

$$\overset{\simeq}{Y}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{R}(x)\bar{q}_{j+1}+\overset{\#}{C}_{12}\overset{\#}{R}(-x)\bar{r}_{j+1}\right] ,$$

$$\overset{\simeq}{F}(x) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{R}(x)\bar{q}_{j+1}+\overset{\#}{C}_{22}\overset{\#}{R}(-x)\bar{r}_{j+1}\right] ,$$

$$\begin{matrix} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{matrix}$$

For $[\ell_{n-1} , \ell_n]$,

$$\overset{\simeq}{Y}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{R}(x)\bar{q}_{n}+\overset{\#}{C}_{12}\overset{\#}{R}(-x)\bar{r}_{n}\right] ,$$

$$\overset{\simeq}{F}(x) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{R}(x)\bar{q}_{n}+\overset{\#}{C}_{12}\overset{\#}{R}(-x)\bar{r}_{n}\right] .$$

<u>APPLYING THE BOUNDARY CONDITION AT $x = 0-0$</u>

$$-\overset{\simeq}{F}(0) = s\,\overset{\#}{Z}_0(0)\,\overset{\simeq}{Y}(0) .$$

If $x = 0$ is substituted in the general solution for $[0 , \ell_1]$ and if $\overset{\#}{R}(0) = \overset{\#}{I}$, then

$$
\left.
\begin{aligned}
\overset{\simeq}{Y}(0) &= \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\bar{q}_1+\overset{\#}{C}_{12}\bar{r}_1\right]\\
\overset{\simeq}{F}(0) &= \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\bar{q}_1+\overset{\#}{C}_{22}\bar{r}_1\right]
\end{aligned}
\right\}
$$

and then substituting back into the boundary condition,

$$
-\overset{\#}{C}_f\left[\overset{\#}{C}_{21}\bar{q}_1+\overset{\#}{C}_{22}\bar{r}_1\right] = s\overset{\#}{Z}_0(0)\,\frac{1}{s}\,\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\bar{q}_1+\overset{\#}{C}_{12}\bar{r}_1\right]
$$

or

$$
-\left[\overset{\#}{C}_{21}\bar{q}_1+\overset{\#}{C}_{22}\bar{r}_1\right] = \overset{\#}{C}{}_f^{-1}\,\overset{\#}{Z}_0(0)\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\bar{q}_1+\overset{\#}{C}_{12}\bar{r}_1\right] \quad .
$$

By direct matrix algebraic operations,

$$
\overset{\#}{z} \equiv \frac{1}{\sqrt{\omega}}\,F\overset{\#}{Z}F \equiv \overset{\#}{C}{}_f^{-1}\,\overset{\#}{Z}\overset{\#}{C}_y \quad , \qquad\qquad F = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\omega} \end{bmatrix}.
$$

Hence,

$$
-\left[\overset{\#}{C}_{21}\bar{q}_1+\overset{\#}{C}_{22}\bar{r}_1\right] = \overset{\#}{z}_0(0)\left[\overset{\#}{C}_{11}\bar{q}_1+\overset{\#}{C}_{12}\bar{r}_1\right]
$$

or

$$
\left[\overset{\#}{C}_{21}+\overset{\#}{z}_0(0)\overset{\#}{C}_{11}\right]\bar{q}_1+\left[\overset{\#}{C}_{22}+\overset{\#}{z}_0(0)\overset{\#}{C}_{12}\right]\bar{r}_1 = 0 \quad .
$$

By definition (see Appendix B),

$$
\left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\overset{\#}{C}_{22}+\overset{\#}{z}_0(0)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\overset{\#}{C}_{12}\right\}\bar{q}_1+\left\{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\overset{\#}{C}_{21}+\overset{\#}{z}_0(0)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\overset{\#}{C}_{11}\right\}\bar{r}_1 = 0 \quad .
$$

By premultiplying $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

$$\left\{\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{C}_{22}+\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{z}_0(0)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{C}_{12}\right\}\overline{q}_1 +$$

$$\left\{\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{C}_{21}+\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{z}_0(0)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{C}_{11}\right\}\overline{r}_1 = 0 \quad .$$

Since

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{A}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \overset{\#}{EAE} = \overset{\Delta}{A} \quad ,$$

$$\left[-\overset{\#}{C}_{22}+\overset{\Delta}{z}_0(0)\overset{\#}{C}_{12}\right]\overline{q}_1+\left[-\overset{\#}{C}_{21}+\overset{\Delta}{z}_0(0)\overset{\#}{C}_{11}\right]\overline{r}_1 = 0 \quad .$$

If the "reflection matrix" (looking to the left) at $x = 0$ is defined as

$$\overset{\#}{\Gamma}_0(0) = \left[-\overset{\#}{C}_{22}+\overset{\Delta}{z}_0(0)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\Delta}{z}_0(0)\overset{\#}{C}_{11}\right] \quad ,$$

then the preceding equation may be rewritten as

$$\overline{q}_1 = \overset{\#}{\Gamma}_0(0)\overline{r}_1$$

which forms the basis for Eqs. (72). If this expression is substituted back into the original general solution for $[0, \ell_1]$,

$$\overset{\simeq}{Y}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{R}(x)(\overset{\#}{\Gamma}_0(0)\overline{r}_1)+\overset{\#}{C}_{12}\overset{\#}{R}(-x)\overline{r}_1\right]$$

or

$$\overset{\simeq}{Y}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{R}(x)\overset{\#}{\Gamma}_0(0)\overset{\#}{R}(x)+\overset{\#}{C}_{12}\right]\overset{\#}{R}(-x)\overline{r}_1 \quad .$$

By defining

$$\overset{\#}{\Gamma}_0(x) = \overset{\#}{R}(x)\overset{\#}{\Gamma}_0(0)\overset{\#}{R}(x) \qquad \text{for } [0, \ell_1] \quad ,$$

then

$$\overset{\simeq}{Y}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{12}\right]\overset{\#}{R}(-x)\overline{r}_1 \quad .$$

Similarly,

$$\tilde{\tilde{F}}(x) = \overset{\#}{C}_f \left[\overset{\#}{C}_{21} \overset{\#}{\Gamma}_0(x) + \overset{\#}{C}_{22} \right] \overset{\#}{R}(-x)\bar{r}_1 \quad .$$

APPLYING BOUNDARY CONDITIONS AT $x = \ell_j$, $j = 1, 2, \ldots, n-1$

If the general solution is substituted directly into

$$\tilde{\tilde{Y}}(\ell_j - 0) = \tilde{\tilde{Y}}(\ell_j + 0) \quad ,$$

then

$$\frac{1}{s}\overset{\#}{C}_y \left[\overset{\#}{C}_{11} \overset{\#}{R}(\ell_j)\bar{q}_j + \overset{\#}{C}_{12} \overset{\#}{R}(-\ell_j)\bar{r}_j \right] = \frac{1}{s}\overset{\#}{C}_y \left[\overset{\#}{C}_{11} \overset{\#}{R}(\ell_j)\bar{q}_{j+1} + \overset{\#}{C}_{12} \overset{\#}{R}(-\ell_j)\bar{r}_{j+1} \right]$$

or

$$\overset{\#}{C}_{11} \overset{\#}{R}(\ell_j)\bar{q}_j + \overset{\#}{C}_{12} \overset{\#}{R}(-\ell_j)\bar{r}_j - \overset{\#}{C}_{11} \overset{\#}{R}(\ell_j)\bar{q}_{j+1} - \overset{\#}{C}_{12} \overset{\#}{R}(-\ell_j)\bar{r}_{j+1} = 0$$

$$j = 1, 2, \ldots, n-1$$

which forms the basis for Eqs. (73). If the general solution is substituted directly into

$$\tilde{\tilde{F}}(\ell_j - 0) - \tilde{\tilde{F}}(\ell_j + 0) = s\overset{\#}{Z}_j(\ell_j)\tilde{\tilde{Y}}(\ell_j) \quad ,$$

then

$$\left[\overset{\#}{C}_f \overset{\#}{C}_{21} - \overset{\#}{Z}_j(\ell_j) \overset{\#}{C}_y \overset{\#}{C}_{11} \right] \overset{\#}{R}(\ell_j)\bar{q}_j + \left[\overset{\#}{C}_f \overset{\#}{C}_{22} - \overset{\#}{Z}_j(\ell_j) \overset{\#}{C}_y \overset{\#}{C}_{12} \right] \overset{\#}{R}(-\ell_j)\bar{r}_j -$$

$$\overset{\#}{C}_f \overset{\#}{C}_{21} \overset{\#}{R}(\ell_j)\bar{q}_{j+1} - \overset{\#}{C}_f \overset{\#}{C}_{22} \overset{\#}{R}(-\ell_j)\bar{r}_{j+1} = 0 \quad .$$

If $\overset{\#}{C}_f^{-1}$ is premultiplied and if $\overset{\#}{C}_f^{-1} \overset{\#}{Z} \overset{\#}{C}_y = \overset{\#}{z}$ is used again, then

$$\left[\overset{\#}{C}_{21} - \overset{\#}{z}_j(\ell_j) \overset{\#}{C}_{11} \right] \overset{\#}{R}(\ell_j)\bar{q}_j - \left[-\overset{\#}{C}_{22} + \overset{\#}{z}_j(\ell_j) \overset{\#}{C}_{12} \right] \overset{\#}{R}(-\ell_j)\bar{r}_j -$$

$$\overset{\#}{C}_{21} \overset{\#}{R}(\ell_j)\bar{q}_{j+1} - \overset{\#}{C}_{22} \overset{\#}{R}(-\ell_j)\bar{r}_{j+1} = 0 \qquad j = 1, 2, \ldots, n-1.$$

This equation forms the basis for Eqs. (74).

<u>APPLYING BOUNDARY CONDITIONS AT $x = \ell_n + 0$</u>

$$\overset{\simeq}{F}(\ell_n) = s\overset{\#}{Z}_n(\ell_n)\overset{\simeq}{Y}(\ell_n) \quad .$$

If $x = \ell_n$ is substituted into the general solution for $[\ell_{n-1}, \ell_n]$, then

$$\left.\begin{aligned} \overset{\simeq}{Y}(\ell_n) &= \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{R}(\ell_n)\bar{q}_n + \overset{\#}{C}_{12}\overset{\#}{R}(-\ell_n)\bar{r}_n\right] \\ \overset{\simeq}{F}(\ell_n) &= \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{R}(\ell_n)\bar{q}_n + \overset{\#}{C}_{22}\overset{\#}{R}(-\ell_n)\bar{r}_n\right] \quad . \end{aligned}\right\}$$

If these expressions are substituted back into the boundary condition equation, then

$$\overset{\#}{C}_{21}\overset{\#}{R}(\ell_n)\bar{q}_n + \overset{\#}{C}_{22}\overset{\#}{R}(-\ell_n)\bar{r}_n = \overset{\#}{z}_n(\ell_n)\left[\overset{\#}{C}_{11}\overset{\#}{R}(\ell_n)\bar{q}_n + \overset{\#}{C}_{12}\overset{\#}{R}(-\ell_n)\bar{r}_n\right]$$

which may be rewritten by solving for $\bar{r}_n$ as:

$$\bar{r}_n = \overset{\#}{R}(\ell_n)\left[-\overset{\#}{C}_{22} + \overset{\#}{z}_n(\ell_n)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21} - \overset{\#}{z}_n(\ell_n)\overset{\#}{C}_{11}\right]\overset{\#}{R}(\ell_n)\bar{q}_n \quad .$$

If the reflection matrix (looking to the right) at $x = \ell_n$ is defined as

$$\overset{\#}{\Gamma}_n(\ell_n) = \left[-\overset{\#}{C}_{22} + \overset{\#}{z}_n(\ell_n)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21} - \overset{\#}{z}_n(\ell_n)\overset{\#}{C}_{11}\right] \quad ,$$

then

$$\bar{r}_n = \overset{\#}{R}(\ell_n)\overset{\#}{\Gamma}_n(\ell_n)\overset{\#}{R}(\ell_n) \quad ,$$

which may be rewritten as

$$\bar{r}_n = \overset{\#}{\Gamma}_n(0)\bar{q}_n$$

by defining

$$\overset{\#}{\Gamma}_n(x) = \overset{\#}{R}(\ell_n - x)\overset{\#}{\Gamma}_n(\ell_n)\overset{\#}{R}(\ell_n - x) \quad .$$

The next to the last equation forms the basis for Eqs. (75). If $\bar{r}_n = \overset{\#}{\Gamma}_n(0)\bar{q}_n$ is substituted back into the original general solution for $[\ell_{n-1}, \ell_n]$, then

$$\overset{\simeq}{Y}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_n(x)\overset{\#}{R}(x)\bar{q}_n\right] \quad .$$

Similarly,

$$\overset{\simeq}{F}(x) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_n(x)\right]\overset{\#}{R}(x)\bar{q}_n \quad .$$

Before going any further, all the equations obtained from boundary conditions of all supports are summarized as follows:

$$\bar{q}_1 = \overset{\#}{\Gamma}_0(0)\bar{r}_1 \tag{72-1}$$

$$\overset{\#}{C}_{11}\overset{\#}{R}(\ell_1)\bar{q}_1+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_1)\bar{r}_1-\overset{\#}{C}_{11}\overset{\#}{R}(\ell_1)\bar{q}_2-\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_1)\bar{r}_2 = 0 \tag{73-1}$$

$$\overset{\#}{C}_{11}\overset{\#}{R}(\ell_2)\bar{q}_2+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_2)\bar{r}_2-\overset{\#}{C}_{11}\overset{\#}{R}(\ell_2)\bar{q}_3-\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_2)\bar{r}_3 = 0 \tag{73-2}$$

$$\vdots \qquad\qquad \vdots$$

$$\overset{\#}{C}_{11}\overset{\#}{R}(\ell_{j-1})\bar{q}_{j-1}+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_{j-1})\bar{r}_{j-1}-\overset{\#}{C}_{11}\overset{\#}{R}(\ell_{j-1})\bar{q}_{j\ell}-\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_{j-1})\bar{r}_{j\ell} = 0 \tag{73-(j-1)}$$

$$\overset{\#}{C}_{11}\overset{\#}{R}(\ell_j)\bar{q}_{jr}+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_j)\bar{r}_{jr}-\overset{\#}{C}_{11}\overset{\#}{R}(\ell_j)\bar{q}_{j+1}-\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_j)\bar{r}_{j+1} = 0 \tag{73-j}$$

$$\vdots \qquad\qquad \vdots$$

$$\overset{\#}{C}_{11}\overset{\#}{R}(\ell_{n-1})\bar{q}_{n-1}+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_{n-1})\bar{r}_{n-1}-\overset{\#}{C}_{11}\overset{\#}{R}(\ell_{n-1})\bar{q}_n-\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_{n-1})\bar{r}_n = 0 \tag{73-(n-1)}$$

$$\left[\overset{\#}{C}_{21}-\overset{\#}{z}_1(\ell_1)\overset{\#}{C}_{11}\right]\overset{\#}{R}(\ell_1)\bar{q}_1-\left[-\overset{\#}{C}_{22}+\overset{\#}{z}_1(\ell_1)\overset{\#}{C}_{12}\right]\overset{\#}{R}(-\ell_1)\bar{r}_1-$$

$$\overset{\#}{C}_{21}\overset{\#}{R}(\ell_1)\bar{q}_2-\overset{\#}{C}_{22}\overset{\#}{R}(-\ell_1)\bar{r}_2=0 \tag{74-1}$$

$$\left[\overset{\#}{C}_{21}-\overset{\#}{z}_2(\ell_2)\overset{\#}{C}_{11}\right]\overset{\#}{R}(\ell_2)\bar{q}_2-\left[-\overset{\#}{C}_{22}+\overset{\#}{z}_2(\ell_2)\overset{\#}{C}_{12}\right]\overset{\#}{R}(-\ell_2)\bar{r}_2-$$

$$\overset{\#}{C}_{21}\overset{\#}{R}(\ell_2)\bar{q}_3-\overset{\#}{C}_{22}\overset{\#}{R}(-\ell_2)\bar{r}_3=0 \tag{74-2}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\left[\overset{\#}{C}_{21}-\overset{\#}{z}_{j-1}(\ell_{j-1})\overset{\#}{C}_{11}\right]\overset{\#}{R}(\ell_{j-1})\bar{q}_{j-1}-\left[-\overset{\#}{C}_{22}+\overset{\#}{z}_{j-1}(\ell_{j-1})\overset{\#}{C}_{12}\right]\overset{\#}{R}(-\ell_{j-1})\bar{r}_{j-1}-$$

$$\overset{\#}{C}_{21}\overset{\#}{R}(\ell_{j-1})\bar{q}_{j\ell}-\overset{\#}{C}_{22}\overset{\#}{R}(-\ell_{j-1})\bar{r}_{j\ell}=0 \tag{74-(j-1)}$$

$$\left[\overset{\#}{C}_{21}-\overset{\#}{z}_j(\ell_j)\overset{\#}{C}_{11}\right]\overset{\#}{R}(\ell_j)\bar{q}_{jr}-\left[-\overset{\#}{C}_{22}+\overset{\#}{z}_j(\ell_j)\overset{\#}{C}_{12}\right]\overset{\#}{R}(-\ell_j)\bar{r}_{jr}-$$

$$\overset{\#}{C}_{21}\overset{\#}{R}(\ell_j)\bar{q}_{j+1}-\overset{\#}{C}_{22}\overset{\#}{R}(-\ell_j)\bar{r}_{j+1}=0 \tag{74-j}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\left[\overset{\#}{C}_{21}-\overset{\#}{z}_{n-1}(\ell_{n-1})\overset{\#}{C}_{11}\right]\overset{\#}{R}(\ell_{n-1})\bar{q}_{n-1}-\left[-\overset{\#}{C}_{22}+\overset{\#}{z}_{n-1}(\ell_{n-1})\overset{\#}{C}_{12}\right]\overset{\#}{R}(-\ell_{n-1})\bar{r}_{n-1}-$$

$$\overset{\#}{C}_{21}\overset{\#}{R}(\ell_{n-1})\bar{q}_n-\overset{\#}{C}_{22}\overset{\#}{R}(-\ell_{n-1})\bar{r}_n=0 \tag{74-(n-1)}$$

$$\bar{r}_n=\overset{\#}{\Gamma}_n(0)\bar{q}_n \tag{75-n}$$

where

$$\overset{\#}{\Gamma}_0(0) = \left[-\overset{\#}{C}_{22}+\overset{\Delta}{z}_0(0)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\Delta}{z}_0(0)\overset{\#}{C}_{11}\right]$$

$$\overset{\#}{\Gamma}_n(\ell_n) = \left[-\overset{\#}{C}_{22}+\overset{\#}{z}_n(\ell_n)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\#}{z}_n(\ell_n)\overset{\#}{C}_{11}\right]$$

$$\overset{\#}{\Gamma}_0(x) = \overset{\#}{R}(x)\overset{\#}{\Gamma}_0(0)\overset{\#}{R}(x)$$

$$\overset{\#}{\Gamma}_n(x) = \overset{\#}{R}(\ell_n-x)\overset{\#}{\Gamma}_n(\ell_n)\overset{\#}{R}(\ell_n-x) \quad .$$

Now, Eqs. (72-1), (73-1), and (74-1) will be considered first. If (72-1) is substituted into (73-1),

$$\underline{\underline{\overset{\#}{R}(-\ell_1)\bar{r}_1}} = \left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(\ell_1)+\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_1)\bar{r}_2\right] \quad .$$

If (72-1) is substituted into (74-1),

$$\left\{\left[\overset{\#}{C}_{21}-\overset{\#}{z}_1(\ell_1)\overset{\#}{C}_{11}\right]\overset{\#}{R}(\ell_1)\overset{\#}{\Gamma}_0(0)\overset{\#}{R}(\ell_1)-\left[-\overset{\#}{C}_{22}+\overset{\#}{z}_1(\ell_1)\overset{\#}{C}_{12}\right]\right\}\underline{\underline{\overset{\#}{R}(-\ell_1)\bar{r}_1}}$$

$$= \overset{\#}{C}_{21}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{22}\overset{\#}{R}(-\ell_1)\bar{r}_2 \quad .$$

Again, if the underlined terms are substituted,

$$\left\{\left[\overset{\#}{C}_{22}+\overset{\#}{C}_{21}\overset{\#}{\Gamma}_0(\ell_1)\right]-\overset{\#}{z}_1(\ell_1)\left[\overset{\#}{C}_{12}+\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(\ell_1)\right]\right.\quad\left[\overset{\#}{C}_{12}+\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(\ell_1)\right]^{-1}$$

$$\times\left[\overset{\#}{C}_{11}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_1)\bar{r}_2\right] = \overset{\#}{C}_{21}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{22}\overset{\#}{R}(-\ell_1)\bar{r}_2 \quad ,$$

or

$$\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\left\{\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(\ell_1)\right]+\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{z}_1(\ell_1)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(\ell_1)\right]\right\}$$

$$\times\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(\ell_1)\right]^{-1}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_1)\bar{r}_2\right]$$

$$= \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\overset{\#}{C}_{22}\overset{\#}{R}(\ell_1)\bar{q}_2+\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\overset{\#}{C}_{21}\overset{\#}{R}(-\ell_1)\bar{r}_2 \quad ,$$

so that

$$\left\{\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(\ell_1)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(\ell_1)\right]^{-1}+\hat{z}_1(\ell_1)\right\}\left[\overset{\#}{C}_{12}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{11}\overset{\#}{R}(-\ell_1)\bar{r}_2\right]$$

$$= \overset{\#}{C}_{22}\overset{\#}{R}(\ell_1)\bar{q}_2 + \overset{\#}{C}_{21}\overset{\#}{R}(-\ell_1)\bar{r}_2 \quad .$$

It will be shown in Appendix D that

$$\hat{z}_{0\ell}(x) = \left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(x)\right]^{-1}$$

or

$$\hat{z}_{0\ell}(\ell_1) = \left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(\ell_1)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(\ell_1)\right]^{-1} \quad .$$

Hence, by substituting,

$$\left[\hat{z}_{0\ell}(\ell_1)+\hat{z}_1(\ell_1)\right]\left[\overset{\#}{C}_{12}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{11}\overset{\#}{R}(-\ell_1)\bar{r}_2\right] = \left[\overset{\#}{C}_{22}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{21}\overset{\#}{R}(-\ell_1)\bar{r}_2\right] \quad .$$

By observing that the total impedance at $x = \ell_1+0$, looking to the left, is

$$\hat{z}_{1\ell}(\ell_1) = \hat{z}_1(\ell_1) + \hat{z}_{0\ell}(\ell_1) \, ,$$

then

$$\hat{z}_{1\ell}(\ell_1)\left[\overset{\#}{C}_{12}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{11}\overset{\#}{R}(-\ell_1)\bar{r}_2\right] = \overset{\#}{C}_{22}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{21}\overset{\#}{R}(-\ell_1)\bar{r}_2 \quad .$$

Solving for $\bar{q}_2$,

$$\bar{q}_2 = \overset{\#}{R}(-\ell_1)\left[-\overset{\#}{C}_{22}+\hat{z}_{1\ell}(\ell_1)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\hat{z}_{1\ell}(\ell_1)\overset{\#}{C}_{11}\right]\overset{\#}{R}(-\ell_1)\bar{r}_2 \quad .$$

By defining the reflection matrix at $x = \ell_1$, looking to the left, as

$$\overset{\#}{\Gamma}_{1\ell}(\ell_1) = \left[-\overset{\#}{C}_{22}+\hat{z}_{1\ell}(\ell_1)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\hat{z}_{1\ell}(\ell_1)\overset{\#}{C}_{11}\right]$$

and

$$\overset{\#}{\Gamma}_{1\ell}(x) = \overset{\#}{R}(x-\ell_1)\overset{\#}{\Gamma}_{1\ell}(\ell_1)\overset{\#}{R}(x-\ell_1) \quad ,$$

the previous equation can be written as

$$\bar{q}_2 = \overset{\#}{\Gamma}_{1\ell}(0)\bar{r}_2 \quad . \tag{72-2}$$

If Eq. (72-2) is substituted back into the general solution for $[\ell_1, \ell_2]$,

$$\tilde{\bar{Y}}(x) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{1\ell}(x)+\overset{\#}{C}_{12}\right]\overset{\#}{R}(-x)\bar{r}_2 \quad .$$

Similarly,

$$\tilde{\bar{F}}(x) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{\Gamma}_{1\ell}(x)+\overset{\#}{C}_{22}\right]\overset{\#}{R}(-x)\bar{r}_2 \; .$$

If Eqs. (72-2), (73-2) and (74-2) are used and if the same mathematical operations as before are followed, the following set of similar expressions can be obtained:

$$\hat{z}_{1\ell}(x) = \left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{1\ell}(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{1\ell}(x)\right]^{-1}$$

$\hat{z}_{2\ell}(\ell_2)$ = total impedance at $x = \ell_2+0$ looking to the left

$$= \hat{z}_2(\ell_2) + \hat{z}_{1\ell}(\ell_2)$$

$\overset{\#}{\Gamma}_{2\ell}(\ell_2)$ = reflection matrix at $x = \ell_2$ looking to the left

$$= \left[-\overset{\#}{C}_{22}+\hat{z}_{2\ell}(\ell_2)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\hat{z}_{2\ell}(\ell_2)\overset{\#}{C}_{11}\right]$$

$$\overset{\#}{\Gamma}_{2\ell}(x) = \overset{\#}{R}(x-\ell_2)\overset{\#}{\Gamma}_{2\ell}(\ell_2)\overset{\#}{R}(x-\ell_2)$$

$$\bar{q}_3 = \overset{\#}{\Gamma}_{2\ell}(0)\bar{r}_3 \tag{72-3}$$

$$\tilde{Y}(x) = \frac{1}{s} \overset{\#}{C}_y \left[\overset{\#}{C}_{11} \overset{\#}{\Gamma}_{2\ell}(x) + \overset{\#}{C}_{12} \right] \overset{\#}{R}(-x) \bar{r}_3$$

$$\tilde{F}(x) = \overset{\#}{C}_f \left[\overset{\#}{C}_{21} \overset{\#}{\Gamma}_{2\ell}(x) + \overset{\#}{C}_{22} \right] \overset{\#}{R}(-x) \bar{r}_3 \quad .$$

Now, if the same mathematical operations are followed again, Eqs. (72-3), (73-3) and (74-3) can be solved; this must be repeated until the solutions of Eqs. (72-(j-1)), (73-(j-1)) and (74-(j-1)) are obtained. These are expressions associated with waves traveling to the left.

Similar mathematical operations can be used in the derivations of expressions associated with waves traveling to the right; however, it is suggested that the right end of the rotating shaft will be considered first. That is, if Eqs. (75-n), (73-(n-1)) and (74-(n-1)) are used and if Eq. (75-n) is substituted into Eq. (73-(n-1)), and so on. The important relationships derived in this appendix are summarized in Chapter 2, Eqs. (15)-(22).

APPENDIX D

MATHEMATICAL DERIVATIONS FOR IMPEDANCES IN TERMS OF REFLECTION MATRICES

The general solution for $[0, \ell_n]$ can be written as

$$\left.\begin{aligned} \overset{\simeq}{Y}(x) &= \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{12}\right]\overset{\#}{R}(-x)\bar{r}_1 && (76)\\ \overset{\simeq}{F}(x) &= \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{22}\right]\overset{\#}{R}(-x)\bar{r}_1 \quad . && (77) \end{aligned}\right\}$$

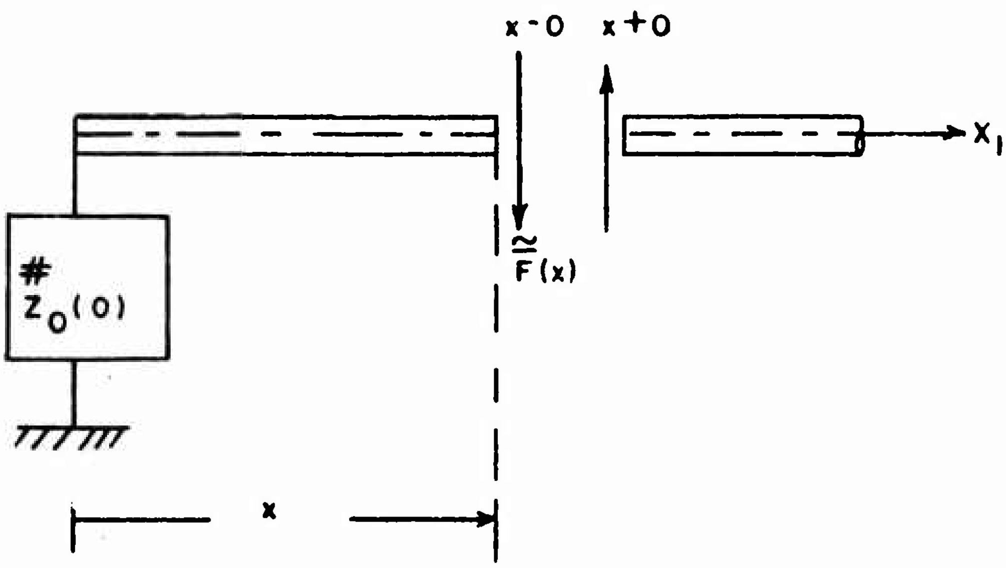

Figure 23. Forces at x on $[0, \ell_1]$.

Since $\overset{\simeq}{F}(x)$ is in the negative direction at $x = x-0$, the impedance relationship at this point can be expressed as (see Figure 23):

$$-\overset{\simeq}{F}(x) = s\overset{\#}{Z}_{0\ell}(x)\overset{\simeq}{Y}(x) \tag{78}$$

where

$\overset{\#}{Z}_{0\ell}(x)$ = total impedance at $x = x-0$ on $[0, \ell_1]$ looking to the left.

From Eq. (76),

$$\overset{\#}{R}(-x)\bar{r}_1 = s\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{12}\right]^{-1}\overset{\#}{C}_y^{-1}\overset{\simeq}{Y}(x)$$

and substituting back into Eq. (77),

$$\overset{\simeq}{F}(x) = s\overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{22}\right]\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{12}\right]^{-1}\overset{\#}{C}_y^{-1}\overset{\simeq}{Y}(x) \quad .$$

If this expression is compared with Eq. (78), it can be concluded that

$$\overset{\#}{Z}_{0\ell}(x) = -\overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{22}\right]\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{12}\right]^{-1}\overset{\#}{C}_y^{-1}$$

or

$$\overset{\#}{z}_{0\ell}(x)\left[\overset{\#}{C}_{21}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{22}\right] = -\left[\overset{\#}{C}_{21}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{22}\right] \quad .$$

If the matrix relationships in Appendix B are used, then

$$\overset{\#}{z}_{0\ell}(x)\left\{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(x)+\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{C}_{11}\right\}$$

$$= -\left\{\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(x) + \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\overset{\#}{C}_{21}\right\} \quad .$$

Premultiplying $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$,

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\overset{\#}{z}_{0\ell}(x)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\left[\overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{11}\right] = -\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\left[\overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(x)+\overset{\#}{C}_{21}\right].$$

Then, on $[0, \ell_1]$,

$$\overset{\triangle}{z}_{0\ell}(x) = \left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(x)\right]^{-1} \quad .$$

If similar mathematical manipulations are followed, the remaining expressions for total impedance on each interval, as listed in Eqs. (23) in Chapter 2, can also be obtained.

One may begin with the right end span, i. e., on $[\ell_{n-1} . \ell_n]$, first, and

recalling that the general solution for $[\ell_{n-1}, \ell_n]$ is

$$\left.\begin{aligned} \overset{\simeq}{Y}(x) &= \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_n(x)\right]\overset{\#}{R}(x)\bar{q}_n && (79)\\ \overset{\simeq}{F}(x) &= \overset{\#}{C}_f\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_n(x)\right]\overset{\#}{R}(x)\bar{q}_n \quad . && (80)\end{aligned}\right\}$$

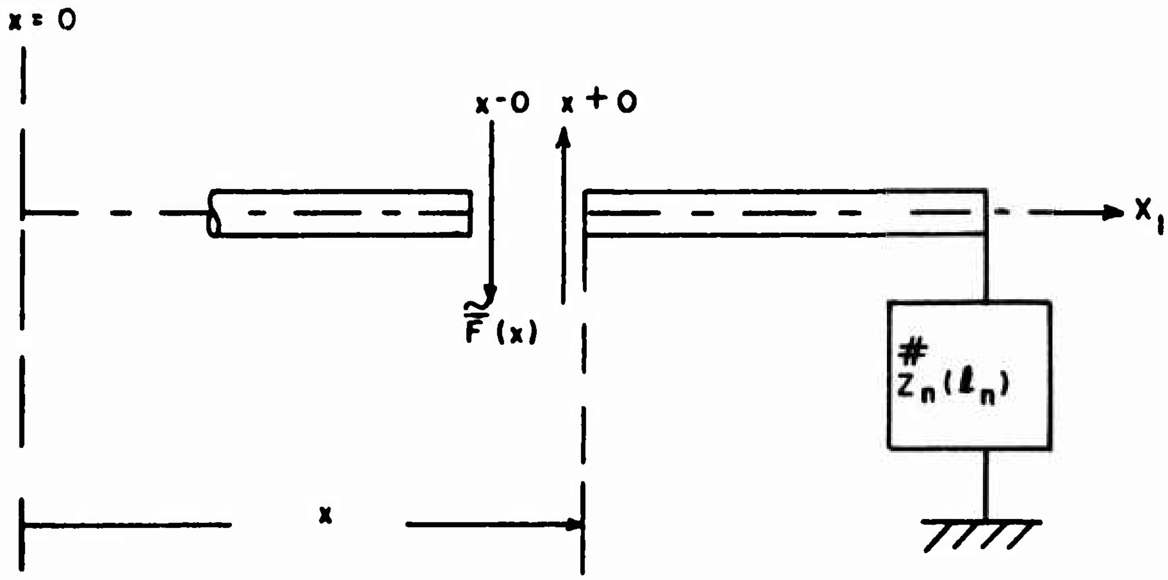

Figure 24. Force at x on $[\ell_{n-1}, \ell_n]$.

Since $\overset{\simeq}{F}(x)$ is in the positive direction at $x = x+0$, the impedance relationship at this point can be expressed as follows (see Figure 24):

$$\overset{\simeq}{F}(x) = s\overset{\#}{Z}_{nr}(x)\overset{\simeq}{Y}(x) \qquad (81)$$

where

$\overset{\#}{Z}_{nr}(x)$ = total impedance at $x = x+0$ on $[\ell_{n-1}, \ell_n]$ looking to the right.

From Eq. (79),

$$\overset{\#}{R}(x)\bar{q}_n = s\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_n(x)\right]^{-1}\overset{\#}{C}_y^{-1}\,\overset{\simeq}{Y}(x)$$

and substituting back into Eq. (80),

$$\tilde{F}(x) = s\overset{\#}{C}_f\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_n(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_n(x)\right]^{-1}\overset{\#}{C}{}_y^{-1}\,\tilde{Y}(x) \quad .$$

When this expression is compared with Eq. (81), it can be concluded that

$$\overset{\#}{Z}_{nr}(x) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_n(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_n(x)\right]^{-1}\overset{\#}{C}{}_y^{-1}$$

or

$$\overset{\#}{z}_{nr}(x) = \overset{\#}{C}{}_f^{-1}\,\overset{\#}{Z}_{nr}(x)\overset{\#}{C}_y = \left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_n(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_n(x)\right]^{-1} \quad .$$

Now, the following equation can be written:

$$\text{On } [\ell_{n-1}, \ell_n] , \qquad \overset{\#}{z}_{nr}(x) = \left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_n(x)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_n(x)\right]^{-1} \quad .$$

If similar mathematical steps are followed, the other expressions of total impedance on each interval, as listed in Eqs. (24) in Chapter 2, also can be obtained.

APPENDIX E

MATHEMATICAL DERIVATIONS FOR THE COMPLETE SOLUTION IN MATRIX FORM

The general solution in Appendix C may be rewritten in matrix form as follows:

On $[0, \ell_1]$,
$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{\Gamma}_0(x)\overset{\#}{R}(-x)\bar{r}_1 \\ \overset{\#}{R}(-x)\bar{r}_1 \end{bmatrix}$$

. . .

. . .

. . .

On $[\ell_{j-2}, \ell_{j-1}]$,
$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{\Gamma}_{(j-2)\ell}(x)\overset{\#}{R}(-x)\bar{r}_{j-1} \\ \overset{\#}{R}(-x)\bar{r}_{j-1} \end{bmatrix}$$

On $[\ell_{j-1}, a_{jk}]$,
$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{\Gamma}_{(j-1)\ell}(x)\overset{\#}{R}(-x)\bar{r}_{k\ell} \\ \overset{\#}{R}(-x)\bar{r}_{k\ell} \end{bmatrix}$$

On $[a_{jk}, \ell_j]$,
$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\bar{q}_{jr} \\ \overset{\#}{\Gamma}_{jr}(x)\overset{\#}{R}(x)\bar{q}_{jr} \end{bmatrix}$$

On $[\ell_j, \ell_{j+1}]$,
$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\bar{q}_{j+1} \\ \overset{\#}{\Gamma}_{(j+1)r}(x)\overset{\#}{R}(x)\bar{q}_{j+1} \end{bmatrix}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots$$

$$\text{On } [\ell_{n-1}, \ell_n] , \quad \begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\bar{q}_n \\ \overset{\#}{\Gamma}_n(x)\overset{\#}{R}(x)\bar{q}_n \end{bmatrix} \tag{82}$$

If the terms in the column matrices in Eqs. (82) are examined, the following relationships are obtained:

$$\begin{aligned} \overset{\#}{\Gamma}_0(x)\overset{\#}{R}(-x)\bar{r}_1 &= \overset{\#}{R}(x)\overset{\#}{\Gamma}_0(0)\overset{\#}{R}(x)\overset{\#}{R}(-x)\bar{r}_1 \\ &= \overset{\#}{R}(x)\overset{\#}{\Gamma}_0(0)\bar{r}_1 \end{aligned}$$

$$\begin{aligned} \overset{\#}{\Gamma}_{1\ell}(x)\overset{\#}{R}(-x)\bar{r}_2 &= \overset{\#}{R}(x-\ell_1)\overset{\#}{\Gamma}_{1\ell}(\ell_1)\overset{\#}{R}(x-\ell_1)\overset{\#}{R}(-x)\bar{r}_2 \\ &= \overset{\#}{R}(x)\overset{\#}{R}(-\ell_1)\overset{\#}{\Gamma}_{1\ell}(\ell_1)\overset{\#}{R}(-\ell_1)\bar{r}_2 \\ &= \overset{\#}{R}(x)\overset{\#}{\Gamma}_{1\ell}(0)\bar{r}_2 \end{aligned}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\overset{\#}{\Gamma}_{(j-1)\ell}(x)\overset{\#}{R}(-x)\bar{r}_{j\ell} = \overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\bar{r}_{j\ell}$$

$$\begin{aligned}
\overset{\#}{\Gamma}_{jr}(x)\overset{\#}{R}(x)\bar{q}_{jr} &= \overset{\#}{R}(\ell_j-x)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-x)\overset{\#}{R}(x)\bar{q}_{jr} \\
&= \overset{\#}{R}(-x)\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j)\bar{q}_{jr} \\
&= \overset{\#}{R}(-x)\overset{\#}{\Gamma}_{jr}(0)\bar{q}_{jr}
\end{aligned}$$

$$\overset{\#}{\Gamma}_{(j+1)r}(x)\overset{\#}{R}(x)\bar{q}_{j+1} = \overset{\#}{R}(-x)\overset{\#}{\Gamma}_{(j+1)r}(0)\bar{q}_{j+1}$$

$$\vdots \qquad\qquad \vdots$$

$$\overset{\#}{\Gamma}_{n}(x)\overset{\#}{R}(x)\bar{q}_{n} = \overset{\#}{R}(-x)\overset{\#}{\Gamma}_{n}(0)\bar{q}_{n} \quad .$$

If the above-derived relationships are used, Eqs. (82) may be rewritten in the matrix forms given by Eqs. (26) in the text.

The integration constants of the matrix relationships given by Eqs. (26) are determined as follows: thus far, the only boundary conditions that have not been used are those at $x = a_{jk}$, the locations of the driving forces. The boundary conditions at these points are as follows (see Figure 8 in Chapter 2):

$$\left.\begin{aligned}
&\tilde{Y}(a_{jk}-0) = \tilde{Y}(a_{jk}+0) && (83) \\
&\tilde{F}(a_{jk}-0) + \tilde{P}(a_{jk}) - \tilde{F}(a_{jk}+0) = 0 && (84)
\end{aligned}\right\} .$$

If $x = a_{jk}$ and is substituted into the corresponding general solution, as given by Eqs. (82), then

$$\left.\begin{aligned}
&\text{at } x = a_{jk}-0\,, \quad \overset{\simeq}{Y}(a_{jk}) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})+\overset{\#}{C}_{12}\right]\overset{\#}{R}(-a_{jk})\bar{r}_{j\ell} \\
&\qquad\qquad\qquad\quad \overset{\simeq}{F}(a_{jk}) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})+\overset{\#}{C}_{12}\right]\overset{\#}{R}(-a_{jk})\bar{r}_{j\ell}
\end{aligned}\right\} \tag{85}$$

$$\left.\begin{aligned}
&\text{and at } x = a_{jk}+0\,, \quad \overset{\simeq}{Y}(a_{jk}) = \frac{1}{s}\overset{\#}{C}_y\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{jr}(a_{jk})\right]\overset{\#}{R}(a_{jk})\bar{q}_{jr} \\
&\qquad\qquad\qquad\qquad\quad \overset{\simeq}{F}(a_{jk}) = \overset{\#}{C}_f\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{jr}(a_{jk})\right]\overset{\#}{R}(a_{jk})\bar{q}_{jr}\;.
\end{aligned}\right\} \tag{86}$$

If Eqs. (85) and (86) are substituted into Eq. (83), then

$$\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})+\overset{\#}{C}_{12}\right]\overset{\#}{R}(-a_{jk})\bar{r}_{j\ell} = \left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{jr}(a_{jk})\right]\overset{\#}{R}(a_{jk})\bar{q}_{jr}\;. \tag{87}$$

Premultiplying $\overset{\#}{C}{}^{-1}_{11}$,

$$\left[\overset{\#}{I}+\overset{\#}{C}{}^{-1}_{11}\,\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{jr}(a_{jk})\right]\overset{\#}{R}(a_{jk})\bar{q}_{jr}-\left[\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})+\overset{\#}{C}{}^{-1}_{11}\,\overset{\#}{C}_{12}\right]\overset{\#}{R}(-a_{jk})\bar{r}_{j\ell} = 0\;. \tag{88}$$

If Eqs. (85) and (86) are substituted into Eq. (84), then

$$\begin{aligned}
&\overset{\#}{C}_f\left[\overset{\#}{C}_{21}\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})+\overset{\#}{C}_{22}\right]\overset{\#}{R}(-a_{jk})\bar{r}_{j\ell}+\overset{\simeq}{P}(a_{jk}) - \\
&\qquad\qquad\qquad\qquad \overset{\#}{C}_f\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{jr}(a_{jk})\right]\overset{\#}{R}(a_{jk})\bar{q}_{jk} = 0\;.
\end{aligned}$$

Premultiplying $\overset{\#}{C}{}^{-1}_{f}$,

$$\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{jr}(a_{jk})\right]\overset{\#}{R}(a_{jk})\bar{q}_{jr}-\left[\overset{\#}{C}_{21}\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})+\overset{\#}{C}_{22}\right]\overset{\#}{R}(-a_{jk})\bar{r}_{j\ell} = \overset{\#}{C}{}^{-1}_{f}\overset{\simeq}{P}(a_{jk})\;. \tag{89}$$

Premultiplying $\overset{\#}{C}{}^{-1}_{21}$,

$$\left[\overset{\#}{I}+\overset{\#}{C}{}_{21}^{-1}\overset{\#}{C}_{22}\overset{\#}{\Gamma}_{jr}(a_{jk})\right]\overset{\#}{R}(a_{jk})\overline{q}_{jr}-\left[\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})+\overset{\#}{C}{}_{21}^{-1}\overset{\#}{C}_{22}\right]\overset{\#}{R}(-a_{jk})\overline{r}_{j\ell}=\overset{\#}{C}{}_{21}^{-1}\overset{\#}{C}{}_{f}^{-1}\overset{\simeq}{P}(a_{jk}) . \tag{90}$$

From Eq. (88) and Eq. (90),

$$\overset{\#}{\Gamma}_{jr}(a_{jk})\overset{\#}{R}(a_{jk})\overline{q}_{jr}-\overset{\#}{R}(-a_{jk})\overline{r}_{j\ell} = -\left[\overset{\#}{C}_{21}\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}-\overset{\#}{C}_{22}\right]^{-1}\overset{\#}{C}{}_{f}^{-1}\overset{\simeq}{P}(a_{jk}) \quad .$$

It may be verified by direct matrix algebraic operations that:

$$\left.\begin{aligned} \overset{\#}{C}_{+} &= \left[\overset{\#}{C}_{21}-\overset{\#}{C}_{22}\overset{\#}{C}{}_{12}^{-1}\overset{\#}{C}_{11}\right]^{-1} \\ \overset{\#}{C}_{-} &= \left[\overset{\#}{C}_{21}\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12} - \overset{\#}{C}_{22}\right]^{-1} \quad . \end{aligned}\right\} \tag{91}$$

Hence,

$$\overset{\#}{R}(-a_{jk})\overline{r}_{j\ell} = \overset{\#}{\Gamma}_{jr}(a_{jk})\overset{\#}{R}(a_{jk})\overline{q}_{jr}+\overset{\#}{C}_{-}\overset{\#}{C}{}_{f}^{-1}\overset{\simeq}{P}(a_{jk}) \quad . \tag{92}$$

From $\overset{\#}{C}{}_{12}^{-1}$ (Eq. (87)),

$$\left[\overset{\#}{C}{}_{12}^{-1}\overset{\#}{C}_{11}+\overset{\#}{\Gamma}_{jr}(a_{jk})\right]\overset{\#}{R}(a_{jk})\overline{q}_{jr}-\left[\overset{\#}{C}{}_{12}^{-1}\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})+\overset{\#}{I}\right]\overline{r}_{j\ell} = 0 \quad . \tag{93}$$

From $\overset{\#}{C}{}_{22}^{-1}$ (Eq. (89)),

$$\left[\overset{\#}{C}{}_{22}^{-1}\overset{\#}{C}_{21}+\overset{\#}{\Gamma}_{jr}(a_{jk})\right]\overset{\#}{R}(a_{jk})\overline{q}_{jr}-\left[\overset{\#}{C}{}_{22}^{-1}\overset{\#}{C}_{21}\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})+\overset{\#}{I}\right]\overset{\#}{R}(-a_{jk})\overline{r}_{j\ell}=\overset{\#}{C}{}_{22}^{-1}\overset{\#}{C}{}_{f}^{-1}\overset{\simeq}{P}(a_{jk}) . \tag{94}$$

From Eq. (93) and Eq. (94),

$$\overset{\#}{R}(a_{jk})\overline{q}_{jr}-\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\overset{\#}{R}(-a_{jk})\overline{r}_{j\ell} =\left[\overset{\#}{C}_{21}-\overset{\#}{C}_{22}\overset{\#}{C}{}_{12}^{-1}\overset{\#}{C}_{11}\right]^{-1}\overset{\#}{C}{}_{f}^{-1}\overset{\simeq}{P}(a_{jk}) \quad .$$

By using relationship (91) again, then

$$\overset{\#}{R}(a_{jk})\overline{q}_{jr} =\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\overset{\#}{R}(-a_{jk})\overline{r}_{j\ell}+\overset{\#}{C}_{+}\overset{\#}{C}{}_{f}^{-1}\overset{\simeq}{P}(a_{jk}) \quad . \tag{95}$$

Now, Eqs. (92) and (95) can be solved simultaneously to yield:

$$\bar{r}_{j\ell} = \overset{\#}{R}(a_{jk})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(a_{jk})\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\right]^{-1}\left[\overset{\#}{\Gamma}_{jr}(a_{jk})\overset{\#}{C}_{+}+\overset{\#}{C}_{-}\right]\overset{\#}{C}{}_{f}^{-1}\overset{\simeq}{P}(a_{jk}) \tag{96}$$

$$\bar{q}_{jr} = \overset{\#}{R}(-a_{jk})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\overset{\#}{\Gamma}_{jr}(a_{jk})\right]^{-1}\left[\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\overset{\#}{C}_{-}+\overset{\#}{C}_{+}\right]\overset{\#}{C}{}_{f}^{-1}\overset{\simeq}{P}(a_{jk}) \quad . \tag{97}$$

If Eqs. (72), (75) are again considered,

$$\left.\begin{array}{ll}
\bar{q}_1 = \overset{\#}{\Gamma}_0(0)\bar{r}_1 & (72\text{-}1) \\
\bar{q}_2 = \overset{\#}{\Gamma}_{1\ell}(0)\bar{r}_2 & (72\text{-}2) \\
\cdot \quad \cdot & \\
\cdot \quad \cdot & \\
\cdot \quad \cdot & \\
\bar{q}_{j-1} = \overset{\#}{\Gamma}_{(j-1)\ell}(0)\bar{r}_{j-1} & (72\text{-}(j\text{-}1)) \\
\bar{q}_{j\ell} = \overset{\#}{\Gamma}_{j\ell}(0)\bar{r}_{j\ell} & (72\text{-}j\ell) \\
\bar{r}_{jr} = \overset{\#}{\Gamma}_{jr}(0)\bar{q}_{jr} & (75\text{-}jr) \\
\bar{r}_{j+1} = \overset{\#}{\Gamma}_{(j+1)r}(0)\bar{q}_{j+1} & (75\text{-}(j+1)) \\
\cdot \quad \cdot & \\
\cdot \quad \cdot & \\
\cdot \quad \cdot & \\
\bar{r}_{n-1} = \overset{\#}{\Gamma}_{(n-1)r}(0)\bar{q}_{n-1} & (75\text{-}(n\text{-}1)) \\
\bar{r}_n = \overset{\#}{\Gamma}_n(0)\bar{q}_n & (75\text{-}n)
\end{array}\right\}$$

and subsequently combined with Eqs. (73), then

$$\bar{r}_1 = \overset{\#}{R}(\ell_1)\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(\ell_1)+\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{R}(\ell_1)\bar{q}_2+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_1)\bar{r}_2\right] \qquad (98\text{-}1)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\bar{r}_{j-1} = \overset{\#}{R}(\ell_{j-1})\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})+\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{R}(\ell_{j-1})\bar{q}_{j\ell}+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_{j-1})\bar{r}_{j\ell}\right] \qquad (98\text{-}(j\text{-}1))$$

$$\bar{q}_{j+1}=\overset{\#}{R}(-\ell_j)\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{R}(\ell_j)\bar{q}_{jr}+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_j)\bar{r}_{jr}\right] \qquad (98\text{-}(j+1))$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\bar{q}_n = \overset{\#}{R}(-\ell_{n-1})\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_n(\ell_{n-1})\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{R}(\ell_{n-1})\bar{q}_{n-1}+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_{n-1})\bar{r}_{n-1}\right]. \qquad (98\text{-}n)$$

If Eq. (72-2) is substituted into Eq. (98-1), then

$$\bar{r}_1 = \overset{\#}{R}(\ell_1)\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_0(\ell_1)+\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{1\ell}(\ell_1)+\overset{\#}{C}_{12}\overset{\#}{R}(-\ell_1)\bar{r}_2\right] .$$

Similarly, if Eq. (72-3) is substituted into Eq. (98-2), then

$$\bar{r}_2 = \overset{\#}{R}(\ell_2)\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{1\ell}(\ell_2)+\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{2\ell}(\ell_2)+\overset{\#}{C}_{12}\right]\overset{\#}{R}(-\ell_2)\bar{r}_3 .$$

If these calculations are continued until Eqs. (72-$j\ell$) and (98-(j-1)) have been reached, the complete set of relationships between integration constants, $\bar{r}$, is obtained.

If Eq. (75-(n-1)) is substituted into Eq. (98-n), then

$$\bar{q}_n = \overset{\#}{R}(-\ell_{n-1})\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_n(\ell_{n-1})\right]^{-1}\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(n-1)r}(\ell_{n-1})\right]\overset{\#}{R}(\ell_{n-1})\bar{q}_{n-1} .$$

Similarly, by substituting Eq. (75-(n-2)) into Eq. (98-(n-1)) and continuing

until Eqs. (75-jr) and (98-(j+1)) have been reached, the complete set of relationships between integration constants, $\bar{q}$, can be obtained.

By combining the previously derived relationships with Eqs. (96) and (97), the complete set of relationships is given as Eqs. (26a) in the text.

APPENDIX F

MATHEMATICAL DERIVATIONS FOR THE COMPLETE SOLUTION IN TRAVELING WAVE FORMS

If the complete solution in matrix form is referenced, i.e., Eqs. (26) and (26a) of Chapter 2, and if the following matrix relationships are used, the solution in traveling wave forms can be rewritten.

The relationships used below are:

$$\left[\overset{\#}{I} - \overset{\#}{A}\right]^{-1} = \overset{\#}{I} + \overset{\#}{A}\left[\overset{\#}{I} - \overset{\#}{A}\right]^{-1} . \tag{99a}$$

$$\left[\overset{\#}{I} - \overset{\#}{A}\overset{\#}{B}\right]^{-1} \overset{\#}{A} = \overset{\#}{A}\left[\overset{\#}{I} - \overset{\#}{B}\overset{\#}{A}\right]^{-1} . \tag{99b}$$

$$\overset{\#}{R}^{-1}(\alpha) = \overset{\#}{R}(-\alpha) . \tag{99c}$$

The expressions for $\overline{r}_{j\ell}$ and $\overline{q}_{jr}$ will be considered first (see Eqs. (26a)); i.e.,

$$\overline{r}_{j\ell} = \overset{\#}{R}(a_{jk})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(a_{jk})\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\right]^{-1}\left[\overset{\#}{\Gamma}_{jr}(a_{jk})\overset{\#}{C}_{+}+\overset{\#}{C}_{-}\right]\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk}) .$$

Substitution of relations of Eqs. (22) into this equation yields

$$\overline{r}_{j\ell} = \overset{\#}{R}(a_{jk})\left[\overset{\#}{I}-\overset{\#}{R}(-a_{jk})\overset{\#}{\Gamma}_{jr}(0)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{R}(a_{jk})\right]^{-1}$$

$$\times\left[\overset{\#}{R}(-a_{jk})\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-a_{jk})\overset{\#}{C}_{+}+\overset{\#}{C}_{-}\right]\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk}) .$$

With relationship (99b), $\overset{\#}{A}\left[\overset{\#}{I}-\overset{\#}{B}\overset{\#}{A}\right]^{-1} = \left[\overset{\#}{I}-\overset{\#}{A}\overset{\#}{B}\right]^{-1}\overset{\#}{A}$, and taking

$$\overset{\#}{A} = \overset{\#}{R}(a_{jk}) , \qquad \overset{\#}{B} = \overset{\#}{R}(-a_{jk})\overset{\#}{\Gamma}_{jr}(0)\overset{\#}{\Gamma}_{(j-1)\ell}(0) ,$$

$$\overline{r}_{j\ell} = \left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(0)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\right]^{-1}\left[\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-a_{jk})\overset{\#}{C}_{+}+\overset{\#}{R}(a_{jk})\overset{\#}{C}_{-}\right]\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk}) . \tag{100}$$

Also from Eqs. (26a),

$$\overline{q}_{jr} = \overset{\#}{R}(-a_{jk})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\overset{\#}{\Gamma}_{jr}(a_{jk})\right]^{-1}\left[\overset{\#}{\Gamma}_{(j-1)\ell}(a_{jk})\overset{\#}{C}_{-}+\overset{\#}{C}_{+}\right]\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk}) .$$

Substitution of Eqs. (21) yields

$$\bar{q}_{jr} = \overset{\#}{R}(-a_{jk})\left[\overset{\#}{I}-\overset{\#}{R}(a_{jk})\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0)\overset{\#}{R}(-a_{jk})\right]^{-1}$$

$$\times\left[\overset{\#}{R}(a_{jk})\overset{\#}{R}(-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(a_{jk}-\ell_{j-1})\overset{\#}{C}_{-}+\overset{\#}{C}_{+}\right]\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk}) \quad .$$

With relationship (99b), $\overset{\#}{A}\left[\overset{\#}{I}-\overset{\#}{B}\overset{\#}{A}\right]^{-1} = \left[\overset{\#}{I}-\overset{\#}{A}\overset{\#}{B}\right]^{-1}\overset{\#}{A}$, and taking

$$\overset{\#}{A} = \overset{\#}{R}(-a_{jk}) \; , \qquad \overset{\#}{B} = \overset{\#}{R}(a_{jk})\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0) \; ,$$

$$\bar{q}_{jr} = \left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0)\right]^{-1}\left[\overset{\#}{R}(-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(a_{jk}-\ell_{j-1})\overset{\#}{C}_{-}+\right.$$

$$\left.\overset{\#}{R}(-a_{jk})\overset{\#}{C}_{+}\right]\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk}) \quad . \tag{101}$$

If Eqs. (100) and (101) are substituted into the last terms of the matrix expressions of Eqs. (26) on $[\ell_{j-1}, a_{jk}]$ and $[a_{jk}, \ell_j]$, the following manipulations can be shown:

$$\overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\bar{r}_{j\ell} = \overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(0)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\right]^{-1}$$

$$\times\left[\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-a_{jk})\overset{\#}{C}_{+}+\overset{\#}{R}(a_{jk})\overset{\#}{C}_{-}\right]\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk}) \quad .$$

With relationship (99b), $\overset{\#}{A}\left[\overset{\#}{I}-\overset{\#}{B}\overset{\#}{A}\right]^{-1} = \left[\overset{\#}{I}-\overset{\#}{A}\overset{\#}{B}\right]^{-1}\overset{\#}{A}$, and taking

$$\overset{\#}{A} = \overset{\#}{\Gamma}_{(j-1)\ell}(0) \; , \qquad \overset{\#}{B} = \overset{\#}{\Gamma}_{jr}(0) \; ,$$

$$\overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\bar{r}_{j\ell} = \overset{\#}{R}(x)\left[\overset{\#}{I}-\overset{\#}{R}(-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(0)\right]^{-1}\overset{\#}{R}(-\ell_{j-1})$$

$$\times\;\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\,\overset{\#}{R}(-\ell_{j-1})\left[\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-a_{jk})\overset{\#}{C}_{+}+\overset{\#}{R}(a_{jk})\overset{\#}{C}_{-}\right]\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk}).$$

With relationship (99b), $\left[\overset{\#}{I}-\overset{\#}{A}\overset{\#}{B}\right]^{-1}\overset{\#}{A} = \overset{\#}{A}\left[\overset{\#}{I}-\overset{\#}{B}\overset{\#}{A}\right]^{-1}$, and taking

$$\overset{\#}{A} = \overset{\#}{R}(-\ell_{j-1}) \, , \qquad \overset{\#}{B} = \overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(0) \, ,$$

$$R(-x)\Gamma_{(j-1)\ell}(0)\overline{r}_{j\ell} = R(x-\ell_{j-1})\left[I-\Gamma_{(j-1)\ell}(\ell_{j-1})\Gamma_{jr}(\ell_{j-1})\right]^{-1}\Gamma_{(j-1)\ell}(\ell_{j-1})$$

$$\times\left[\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-a_{jk})\overset{\#}{C}_+ + \overset{\#}{R}(a_{jk}-\ell_{j-1})\overset{\#}{C}_-\right]\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{jk}) \, . \qquad (102)$$

Also,

$$\overset{\#}{R}(-x)r_{j\ell} = \left\{ \overset{\#}{R}(-x)\left[\overset{\#}{I}-\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\right]^{-1}\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-a_{jk})\overset{\#}{C}_+ + \right.$$

$$\left. \overset{\#}{R}(-x)\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(0)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\right]^{-1}\overset{\#}{R}(a_{jk})\overset{\#}{C}_- \right\}\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{jk}) \quad .$$

With relationship (99b), $\left[\overset{\#}{I}\text{-}\overset{\#}{A}\overset{\#}{B}\right]^{-1}\overset{\#}{A} = \overset{\#}{A}\left[\overset{\#}{I}\text{-}\overset{\#}{B}\overset{\#}{A}\right]^{-1}$, and taking

$$\overset{\#}{A} = \overset{\#}{R}(\ell_j) \, , \qquad \overset{\#}{B} = \overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(0) \, ,$$

$$\overset{\#}{R}(-x)\overline{r}_{j\ell} = \left\{ \overset{\#}{R}(-x)\overset{\#}{R}(\ell_j)\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{R}(\ell_j)\right]^{-1}\overset{\#}{\Gamma}_{jr}(\ell_j) \right.$$

$$\left. \times\overset{\#}{R}(\ell_j-a_{jk})\overset{\#}{C}_+ + \overset{\#}{R}(-x)\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(0)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\right]^{-1}\overset{\#}{R}(a_{jk})\overset{\#}{C}_- \right\}\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{jk}) \quad .$$

With relationship (99a), $\left[\overset{\#}{I} - \overset{\#}{A}\right]^{-1} = \overset{\#}{I} + \overset{\#}{A}\left[\overset{\#}{I} - \overset{\#}{A}\right]^{-1}$, and taking

$$\overset{\#}{A} = \overset{\#}{\Gamma}_{jr}(0)\overset{\#}{\Gamma}_{(j-1)\ell}(0) \, ,$$

$$\overset{\#}{R}(-x)\overline{r}_{j\ell} = \left\{ \overset{\#}{R}(\ell_j-x)\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\right]^{-1}\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-a_{jk})\overset{\#}{C}_+ + \overset{\#}{R}(a_{jk}-x)\overset{\#}{C}_- + \right.$$

$$\overset{\#}{R}(-x)\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\left[\overset{\#}{I}-\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\right]^{-1}$$

$$\left. \times\overset{\#}{R}(a_{jk})\overset{\#}{C}_- \right\}\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{jk}) \quad .$$

With relationship (99b), $\overset{\#}{A}\left[\overset{\#}{I}\text{-}\overset{\#}{B}\overset{\#}{A}\right]^{-1} = \left[\overset{\#}{I}\text{-}\overset{\#}{A}\overset{\#}{B}\right]^{-1}\overset{\#}{A}$, and taking

$$\overset{\#}{A} = \overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(0) \,, \qquad \overset{\#}{B} = \overset{\#}{R}(\ell_j) \,,$$

$$\begin{aligned}\overset{\#}{R}(-x)\bar{r}_{j\ell} = \Big\{ & \overset{\#}{R}(\ell_j - x)\Big[\overset{\#}{I} - \overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\Big]^{-1}\overset{\#}{\Gamma}_{jr}(\ell_j)\Big[\overset{\#}{R}(\ell_j - a_{jk})\overset{\#}{C}_+ + \\ & \overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(a_{jk}-\ell_{j-1})\overset{\#}{C}_-\Big] + \overset{\#}{R}(a_{jk}-x)\overset{\#}{C}_-\Big\}\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{jk}) \,. \qquad (103)\end{aligned}$$

Also,

$$\begin{aligned}\overset{\#}{R}(x)\bar{q}_{jr} = \overset{\#}{R}(x)\Big\{ & \Big[\overset{\#}{I} - \overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0)\Big]^{-1}\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{R}(a_{jk})\overset{\#}{C}_- + \\ & \Big[\overset{\#}{I} - \overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0)\Big]^{-1}\overset{\#}{R}(-a_{jk})\overset{\#}{C}_+\Big\}\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{jk}) \,.\end{aligned}$$

With relationship (99a), $\Big[\overset{\#}{I} - \overset{\#}{A}\Big]^{-1} = \overset{\#}{I} + \overset{\#}{A}\Big[\overset{\#}{I} - \overset{\#}{A}\Big]^{-1}$, and taking

$$\overset{\#}{A} = \overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0) \,,$$

$$\begin{aligned}\overset{\#}{R}(x)\bar{q}_{jr} = \overset{\#}{R}(x)\Big\{ & \Big[\overset{\#}{I} - \overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0)\Big]^{-1}\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{R}(a_{jk})\overset{\#}{C}_- + \overset{\#}{R}(-a_{jk})\overset{\#}{C}_+ + \\ & \overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0)\Big[\overset{\#}{I} - \overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0)\Big]^{-1}\overset{\#}{R}(-a_{jk})\overset{\#}{C}_+\Big\}\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{jk}) \,.\end{aligned}$$

With relationship (99b), $\overset{\#}{A}\Big[\overset{\#}{I} - \overset{\#}{B}\overset{\#}{A}\Big]^{-1} = \Big[\overset{\#}{I} - \overset{\#}{A}\overset{\#}{B}\Big]^{-1}\overset{\#}{A}$, and taking

$$\overset{\#}{A} = \overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0) \,, \qquad \overset{\#}{B} = \overset{\#}{I} \,,$$

$$\begin{aligned}\overset{\#}{R}(x)\bar{q}_{jr} = \Big\{ & \overset{\#}{R}(x)\Big[\overset{\#}{I} - \overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{\Gamma}_{jr}(0)\Big]^{-1}\overset{\#}{\Gamma}_{(j-1)\ell}(0)\Big[\overset{\#}{R}(a_{jk})\overset{\#}{C}_- + \\ & \overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j - a_{jk})\overset{\#}{C}_+\Big] + \overset{\#}{R}(x-a_{jk})\overset{\#}{C}_+\Big\}\overset{\#}{C}{}_f^{-1}\overset{\simeq}{P}(a_{jk}) \,.\end{aligned}$$

By following the same matrix algebraic operations as those used in the evaluations of $\overset{\#}{R}(-x)\bar{r}_{j\ell}$, the preceding equation can be transformed to:

$$\overset{\#}{R}(x)\overline{q}_{jr} = \left\{ \overset{\#}{R}(x-\ell_{j-1}) \left[\overset{\#}{I} - \overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1}) \overset{\#}{\Gamma}_{jr}(\ell_{j-1}) \right]^{-1} \overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1}) \right.$$

$$\times \left[\overset{\#}{R}(a_{jk}-\ell_{j-1})\overset{\#}{C}_- + \overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j-a_{jk})\overset{\#}{C}_+ \right] +$$

$$\left. \overset{\#}{R}(x-a_{jk})\overset{\#}{C}_+ \right\} \overset{\#}{C}_f^{-1} \overset{\simeq}{P}(a_{jk}) \quad . \tag{104}$$

Also,

$$\overset{\#}{R}(-x)\overset{\#}{\Gamma}_{jr}(0)\overline{q}_{jr} = \overset{\#}{R}(-x)\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j) \left[\overset{\#}{I} - \overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{R}(\ell_j)\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j) \right]^{-1}$$

$$\times \left[\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{R}(a_{jk})\overset{\#}{C}_- + \overset{\#}{R}(-a_{jk})\overset{\#}{C}_+ \right] \overset{\#}{C}_f^{-1} \overset{\simeq}{P}(a_{jk}) \quad .$$

With relationship (99b), $\overset{\#}{A}\left[\overset{\#}{I}-\overset{\#}{B}\overset{\#}{A}\right]^{-1} = \left[\overset{\#}{I}-\overset{\#}{A}\overset{\#}{B}\right]^{-1}\overset{\#}{A}$, and taking

$$\overset{\#}{A} = \overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{R}(\ell_j) \ , \qquad \overset{\#}{B} = \overset{\#}{\Gamma}_{(j-1)\ell}(0)\overset{\#}{R}(\ell_j) \ ,$$

$$\overset{\#}{R}(-x)\overset{\#}{\Gamma}_{jr}(0)\overline{q}_{jr} = \overset{\#}{R}(\ell_j-x) \left[\overset{\#}{I} - \overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j) \right]^{-1} \overset{\#}{\Gamma}_{jr}(\ell_j)$$

$$\times \left[\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{R}(a_{jk}-\ell_{j-1})\overset{\#}{C}_- + \overset{\#}{R}(\ell_j-a_{jk})\overset{\#}{C}_+ \right] \overset{\#}{C}_f^{-1} \overset{\simeq}{P}(a_{jk}) \ . \tag{105}$$

Eqs. (102), (103), (104), and (105) are the complex entries of the right-hand column matrix of Eqs. (26),

On $[\ell_{j-1}, a_{jk}]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overline{r}_{j\ell} \\ R(-x)\overline{r}_{j\ell} \end{bmatrix}$$

On $[a_{jk}, \ell_j]$,

$$\begin{bmatrix} \tilde{\overline{Y}} \\ \\ \tilde{\overline{F}} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\overline{q}_{jr} \\ \overset{\#}{R}(-x)\overset{\#}{\Gamma}_{jr}(0)\overline{q}_{jr} \end{bmatrix} .$$

Also, if the identity of the various parts of Eqs. (102), (103), (104), and (105) is observed, the following relationships can be obtained:

$$\overset{\#}{R}(-x)\overline{r}_{j\ell} = \overset{\#}{R}(-x)\overset{\#}{\Gamma}_{jr}(0)\overline{q}_{jr}+\overset{\#}{R}(a_{jk}-x)\overset{\#}{C}_-\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk}) \tag{106}$$

$$\overset{\#}{R}(x)\overline{q}_{jr} = \overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overline{r}_{j\ell}+\overset{\#}{R}(x-a_{jk})\overset{\#}{C}_+\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk}) \quad . \tag{107}$$

From the above-derived equations, some useful functions related to the concept of traveling waves can be defined. On $[\ell_{j-1}, \ell_j]$, the incident waves, caused by the single driving force $\overset{\simeq}{P}(a_{jk})$, traveling to the right and to the left, respectively, are as follows:

$$\overset{>}{\overline{U}}_{jjk}(x) = \overset{\#}{R}(x-a_{jk})\overset{\#}{C}_+\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk})[H(x-a_{jk})-H(x-\ell_j)] \tag{108}$$

exists only on $[a_{jk}, \ell_j]$

$$\overset{<}{\overline{U}}_{jjk}(x) = \overset{\#}{R}(a_{jk}-x)\overset{\#}{C}_-\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk})[H(a_{jk}-x)-H(\ell_{j-1}-x)] \tag{109}$$

exists only on $[\ell_{j-1}, a_{jk}]$.

If the expressions for incident waves are used, Eqs. (102), (103), (104), and (105) can be rewritten as follows:

On $[\ell_{j-1}, a_{jk}]$,

$$\overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\overline{r}_{j\ell}=\overset{\#}{R}(x-\ell_{j-1})\left[\overset{\#}{I}-\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\right]^{-1}\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})$$
$$\times\left[\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{>}{\overline{U}}_{jjk}(\ell_j)+\overset{<}{\overline{U}}_{jjk}(\ell_{j-1})\right] \tag{110}$$

$$\overset{\#}{R}(-x)\overline{r}_{j\ell}=\overset{\#}{R}(\ell_j-x)\left[I-\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\right]^{-1}\overset{\#}{\Gamma}_{jr}(\ell_j)$$

$$\times\left[\overset{>}{\overline{U}}_{jjk}(\ell_j)+\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{<}{\overline{U}}_{jjk}(\ell_{j-1})\right]+\overset{<}{\overline{U}}_{jjk}(x)\,. \tag{111}$$

On $[a_{jk}, \ell_j]$,

$$\overset{\#}{R}(x)\overline{q}_{jr}=\overset{\#}{R}(x-\ell_{j-1})\left[I-\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\right]^{-1}\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})$$

$$\times\left[\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{>}{\overline{U}}_{jjk}(\ell_j)+\overset{<}{\overline{U}}_{jjk}(\ell_{j-1})\right]+\overset{>}{\overline{U}}_{jjk}(x) \tag{112}$$

$$\overset{\#}{R}(-x)\overset{\#}{\Gamma}_{jr}(0)\overline{q}_{jr}=\overset{\#}{R}(\ell_j-x)\left[I-\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\right]^{-1}\overset{\#}{\Gamma}_{jr}(\ell_j)$$

$$\times\left[\overset{>}{\overline{U}}_{jjk}(\ell_j)+\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{<}{\overline{U}}_{jjk}(\ell_{j-1})\right]\,. \tag{113}$$

From an examination of the above expressions, the reflected waves, originated from the single driving force $\overset{\simeq}{P}(a_{jk})$, traveling to the right and to the left, respectively, can be defined as follows:

$$\overset{\gg}{\overline{U}}_{jjk}(x)=\overset{\#}{R}(x-\ell_{j-1})\left[I-\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_{j-1})\right]^{-1}\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})$$

$$\times\left[\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{>}{\overline{U}}_{jjk}(\ell_j)+\overset{<}{\overline{U}}_{jjk}(\ell_{j-1})\right]\left[H(x-\ell_{j-1})-H(x-\ell_j)\right] \tag{114}$$

$$\overset{\ll}{\overline{U}}_{jjk}(x)=\overset{\#}{R}(\ell_j-x)\left[I-\overset{\#}{\Gamma}_{jr}(\ell_j)\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_j)\right]^{-1}\overset{\#}{\Gamma}_{jr}(\ell_j)$$

$$\times\left[\overset{>}{\overline{U}}_{jjk}(\ell_j)+\overset{\#}{R}(\ell_j-\ell_{j-1})\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})\overset{<}{\overline{U}}_{jjk}(\ell_{j-1})\right]\left[H(x-\ell_{j-1})-H(x-\ell_j)\right] \tag{115}$$

both exist only on $[\ell_{j-1}, \ell_j]$.

When the expressions for reflected waves are used, Eqs. (110), (111), (112), and (113) can be rewritten in much more compact forms as follows:

On $[\ell_{j-1}, a_{jk}]$,

$$\overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-1)\ell}(0)\bar{r}_{j\ell} = \overset{\gg}{\bar{U}}_{jjk}(x) \tag{116}$$

$$\overset{\#}{R}(-x)\bar{r}_{j\ell} = \overset{\ll}{\bar{U}}_{jjk}(x) + \overset{<}{\bar{U}}_{jjk}(x) . \tag{117}$$

On $[a_{jk}, \ell_j]$,

$$\overset{\#}{R}(x)\bar{q}_{jr} = \overset{\gg}{\bar{U}}_{jjk}(x) + \overset{>}{\bar{U}}_{jjk}(x) \tag{118}$$

$$\overset{\#}{R}(-x)\overset{\#}{\Gamma}_{jr}(0)\bar{q}_{jr} = \overset{\ll}{\bar{U}}_{jjk}(x) . \tag{119}$$

Now, if the above expressions are substituted back into the matrix form, the dynamic response on $[\ell_{j-1}, \ell_j]$ can be expressed as follows:

On $[\ell_{j-1}, a_{jk}]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\gg}{\bar{U}}_{jjk}(x) \\ \overset{\ll}{\bar{U}}_{ijk}(x)+\overset{<}{\bar{U}}_{jjk}(x) \end{bmatrix} . \tag{120}$$

On $[a_{jk}, \ell_j]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\gg}{\bar{U}}_{jjk}(x)+\overset{>}{\bar{U}}_{jjk}(x) \\ \overset{\ll}{\bar{U}}_{jjk}(x) \end{bmatrix} . \tag{121}$$

From the following observations,

$\overset{>}{\bar{U}}_{jjk}(x)$ exists only on $[a_{jk}, \ell_j]$

and $\overset{<}{\bar{U}}_{jjk}(x)$ exists only on $[\ell_{j-1}, a_{jk}]$,

additional terms may be included in the preceding matrix expressions without changing their original nature, as follows:

On $[\ell_{j-1}, a_{jk}]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\gg}{\overline{U}}_{jjk}(x)+\overset{>}{\overline{U}}_{jjk}(x) \\ \overset{\ll}{\overline{U}}_{jjk}(x)+\overset{<}{\overline{U}}_{jjk}(x) \end{bmatrix} . \tag{122}$$

On $[a_{jk}, \ell_j]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\gg}{\overline{U}}_{jjk}(x)+\overset{>}{\overline{U}}_{jjk}(x) \\ \overset{\ll}{\overline{U}}_{jjk}(x)+\overset{<}{\overline{U}}_{jjk}(x) \end{bmatrix} . \tag{123}$$

It is obvious that the above two matrix equations are completely identical; hence, they may be combined as follows:

On $[\ell_{j-1}, \ell_j]$,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\gg}{\overline{U}}_{jjk}(x)+\overset{>}{\overline{U}}_{jjk}(x) \\ \overset{\ll}{\overline{U}}_{jjk}(x)+\overset{<}{\overline{U}}_{jjk}(x) \end{bmatrix} . \tag{124}$$

This matrix equation suggests that the total traveling waves, originated from the single driving force $\overset{\simeq}{P}(a_{jk})$, traveling to the right and to the left, respectively, are as follows:

$$\overset{\ggg}{\overline{U}}_{jjk}(x) = \overset{\gg}{\overline{U}}_{jjk}(x) + \overset{>}{\overline{U}}_{jjk}(x) \tag{125}$$

and

$$\overset{\lll}{\overline{U}}_{jjk}(x) = \overset{\ll}{\overline{U}}_{jjk}(x) + \overset{<}{\overline{U}}_{jjk}(x) . \tag{126}$$

Hence, the final expression of the dynamic response caused by the single driving force $\overset{\simeq}{P}(a_{jk})$ on $[\ell_{j-1}, \ell_j]$ is as follows:

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\overline{U}}_{jjk}(x) \\ \overset{\lll}{\overline{U}}_{jjk}(x) \end{bmatrix} . \tag{127}$$

If the superposition technique is used, the dynamic response on $[\ell_{j-1}, \ell_j]$ caused by all the driving forces, $\overset{\simeq}{P}(a_{jk})$, $k = 1, 2, \ldots, k(j)$, in this span is as follows:

Incident waves on $[\ell_{j-1}, \ell_j]$:

$$\overset{>}{\overline{U}}_{jj}(x) = \sum_{k=1}^{k(j)} \overset{\#}{R}(x-a_{jk})\overset{\#}{C}_+\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk})\left[H(x-a_{jk})-H(x-\ell_j)\right] \tag{128}$$

and

$$\overset{<}{\overline{U}}_{jj}(x) = \sum_{k=1}^{k(j)} \overset{\#}{R}(a_{jk}-x)\overset{\#}{C}_-\overset{\#}{C}_f^{-1}\overset{\simeq}{P}(a_{jk})\left[H(a_{jk}-x)-H(\ell_{j-1}-x)\right] . \tag{129}$$

Reflected waves on $[\ell_{j-1}, \ell_j]$:

$$\overset{\gg}{\overline{U}}_{jj}(x) = \sum_{k=1}^{k(j)} \overset{\gg}{\overline{U}}_{jjk}(x) \tag{130}$$

and

$$\overset{\ll}{\overline{U}}_{jj}(x) = \sum_{k=1}^{k(j)} \overset{\ll}{\overline{U}}_{jjk}(x) . \tag{131}$$

Total waves on $[\ell_{j-1}, \ell_j]$:

$$\overset{\ggg}{\overline{U}}_{jj}(x) = \sum_{k=1}^{k(j)} \overset{\ggg}{\overline{U}}_{jjk}(x) = \sum_{k=1}^{k(j)} \overset{\gg}{\overline{U}}_{jjk}(x)+\overset{>}{\overline{U}}_{jjk}(x) \tag{132}$$

and

$$\overset{\lll}{\overline{U}}_{jj}(x) = \sum_{k=1}^{k(j)} \overset{\lll}{\overline{U}}_{jjk}(x) = \sum_{k=1}^{k(j)} \overset{\ll}{\overline{U}}_{jjk}(x)+\overset{<}{\overline{U}}_{jjk}(x) \quad . \tag{133}$$

The solution of the dynamic response on $[\ell_{j-1}, \ell_j]$ is as follows:

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\overline{U}}_{jj}(x) \\ \overset{\lll}{\overline{U}}_{jj}(x) \end{bmatrix} . \tag{134}$$

Now, the traveling wave forms in other spans of the shaft will be investigated.

On $[\ell_{j-2}, \ell_{j-1}]$ (see Eqs. (26)),

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-2)\ell}(0)\overline{r}_{j-1} \\ \overset{\#}{R}(-x)\overline{r}_{j-1} \end{bmatrix} .$$

From Eq. (26a), the following equation can be obtained by substitution:

$$\overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-2)\ell}(0)\overline{r}_{j-1} = \overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-2)\ell}(0) \left\{ \overset{\#}{R}(\ell_{j-1}) \left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})+\overset{\#}{C}_{12}\right]^{-1} \right.$$
$$\left. \times \left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})+\overset{\#}{C}_{12}\right] \overset{\#}{R}(-\ell_{j-1})\overline{r}_{j\ell} \right\} .$$

From Eq. (82),

$$\overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-2)\ell}(0)r_{j-1} = \overset{\#}{R}(x-\ell_{j-2})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})\overset{\#}{R}(\ell_{j-1}-\ell_{j-2})$$
$$\times \left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})+\overset{\#}{C}_{12}\right]^{-1} \left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})+\overset{\#}{C}_{12}\right]$$
$$\times s\left[\overset{\#}{C}_{11}\overset{\#}{\Gamma}_{(j-1)\ell}(\ell_{j-1})+\overset{\#}{C}_{12}\right]^{-1}\overset{\#}{C}_y^{-1}\overset{\simeq}{Y}(\ell_{j-1}+0) .$$

From Eq. (134), and since $\overset{\#}{C}{}^{-1}_{12}\overset{\#}{C}_{11} = \overset{\#}{C}{}^{-1}_{11}\overset{\#}{C}_{12}$,

$$\overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-2)\ell}(0)\bar{r}_{j-1}$$

$$=\overset{\#}{R}(x-\ell_{j-2})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})\overset{\#}{R}(\ell_{j-1}-\ell_{j-2})\left[\overset{\#}{I}+\overset{\#}{C}{}^{-1}_{11}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})\right]^{-1}$$

$$\times\left[\overset{\#}{C}{}^{-1}_{11}\overset{\#}{C}_{12}\overset{\ggg}{\bar{U}}_{jj}(\ell_{j-1})+\overset{\lll}{\bar{U}}_{jj}(\ell_{j-1})\right] .$$

The total wave traveling to the right on $[\ell_{j-2}, \ell_{j-1}]$, i.e., the $(j-1)^{th}$ span, can be defined as follows:

$$\overset{\ggg}{\bar{U}}_{(j-1)j}(x) = \overset{\#}{R}(x)\overset{\#}{\Gamma}_{(j-2)\ell}(0)\bar{r}_{j-1}$$

or

$$= \overset{\#}{R}(x-\ell_{j-2})\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-2})\overset{\#}{R}(\ell_{j-1}-\ell_{j-2})\left[\overset{\#}{I}+\overset{\#}{C}{}^{-1}_{11}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})\right]^{-1}$$

$$\times\left[\overset{\#}{C}{}^{-1}_{11}\overset{\#}{C}_{12}\overset{\ggg}{\bar{U}}_{jj}(\ell_{j-1})+\overset{\lll}{\bar{U}}_{jj}(\ell_{j-1})\right] . \tag{135}$$

Similarly, the total wave traveling to the left on $[\ell_{j-2}, \ell_{j-1}]$, i.e., the $(j-1)^{th}$ span, can be obtained as follows:

$$\overset{\lll}{\bar{U}}_{(j-1)j}(x) = \overset{\#}{R}(-x)\bar{r}_{j-1}$$

$$=\overset{\#}{R}(\ell_{j-1}-x)\left[\overset{\#}{I}+\overset{\#}{C}{}^{-1}_{11}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-2)\ell}(\ell_{j-1})\right]^{-1}\left[\overset{\#}{C}{}^{-1}_{11}\overset{\#}{C}_{12}\overset{\ggg}{\bar{U}}_{jj}(\ell_{j-1})+\overset{\lll}{\bar{U}}_{jj}(\ell_{j-1})\right] . \tag{136}$$

Hence, the dynamic response on $[\ell_{j-2}, \ell_{j-1}]$, i.e., the $(j-1)^{th}$ span, caused by driving forces $\overset{\simeq}{P}(a_{jk})$, $k = 1, 2, \ldots, k(j)$, on $[\ell_{j-1}, \ell_j]$, i.e., the j^{th} span, can be expressed by the following matrix equation:

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix}\begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix}\begin{bmatrix} \overset{\ggg}{\bar{U}}_{(j-1)j}(x) \\ \overset{\lll}{\bar{U}}_{(j-1)j}(x) \end{bmatrix} . \tag{137}$$

With the same mathematical manipulations, the following set of matrix equations can be obtained:

On $[\ell_{j-3}, \ell_{j-2}]$, i.e., the $(j-2)^{th}$ span,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\overline{U}}_{(j-2)j}(x) \\ \overset{\lll}{\overline{U}}_{(j-1)j}(x) \end{bmatrix}$$

where

$$\overset{\lll}{\overline{U}}_{(j-2)j}(x)=\overset{\#}{R}(\ell_{j-2}-x)\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j-3)\ell}(\ell_{j-2})\right]^{-1}$$

$$\times\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\overline{U}}_{(j-1)j}(\ell_{j-2})+\overset{\lll}{\overline{U}}_{(j-1)j}(\ell_{j-2})\right] .$$

On $[\ell_{j-4}, \ell_{j-3}]$, i.e., the $(j-3)^{th}$ span,

$$\begin{matrix} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{matrix}$$

$$\overset{\lll}{\overline{U}}_{3j}(x)= \overset{\#}{R}(\ell_3-x)\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{2\ell}(\ell_3)\right]^{-1}\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\overline{U}}_{4j}(\ell_3)+\overset{\lll}{\overline{U}}_{4j}(\ell_3)\right] .$$

On $[\ell_1, \ell_2]$, i.e., the 2^{nd} span,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\overline{U}}_{2j}(x) \\ \overset{\lll}{\overline{U}}_{2j}(x) \end{bmatrix}$$

where

$$\overset{\lll}{\overline{U}}_{2j}(x)=\overset{\#}{R}(\ell_2-x)\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{1\ell}(\ell_2)\right]^{-1}\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\overline{U}}_{3j}(\ell_2)+\overset{\lll}{\overline{U}}_{3j}(\ell_2)\right] .$$

On $[0, \ell_1]$, i.e., the 1^{st} span,

$$\begin{bmatrix}\overset{\simeq}{Y}\\ \overset{\simeq}{F}\end{bmatrix} = \begin{bmatrix}\frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0}\\ \overset{\#}{0} & \overset{\#}{C}_f\end{bmatrix}\begin{bmatrix}\overset{\#}{C}_{11} & \overset{\#}{C}_{12}\\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22}\end{bmatrix}\begin{bmatrix}\overset{\ggg}{\overline{U}}_{1j}(x)\\ \overset{\lll}{\overline{U}}_{1j}(x)\end{bmatrix}$$

where

$$\overset{\lll}{\overline{U}}_{1j}(x)=\overset{\#}{R}(\ell_1-x)\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(\ell_1)\right]^{-1}\left[\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ggg}{\overline{U}}_{2j}(\ell_1)+\overset{\lll}{\overline{U}}_{2j}(\ell_1)\right] .$$

Similar mathematical manipulations can be applied to other portions of the shaft.

On $[\ell_j, \ell_{j-1}]$ (see Eq. (26)),

$$\begin{bmatrix}\overset{\simeq}{Y}\\ \overset{\simeq}{F}\end{bmatrix} = \begin{bmatrix}\frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0}\\ \overset{\#}{0} & \overset{\#}{C}_f\end{bmatrix}\begin{bmatrix}\overset{\#}{C}_{11} & \overset{\#}{C}_{12}\\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22}\end{bmatrix}\begin{bmatrix}\overset{\#}{R}(x)\overline{q}_{j+1}\\ \overset{\#}{R}(-x)\overset{\#}{\Gamma}_{(j+1)r}(0)\overline{q}_{j+1}\end{bmatrix} .$$

From Eq. (26a), the following equation can be obtained by substitution:

$$\overset{\#}{R}(x)\overline{q}_{j+1}=\overset{\#}{R}(x)\left\{\overset{\#}{R}(-\ell_j)\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}\right.$$
$$\left.\times\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{jr}(\ell_j)\right]\overset{\#}{R}(\ell_j)\overline{q}_{jr}\right\} .$$

From Eqs. (82),

$$\overset{\#}{R}(x)\overline{q}_{j+1}=\overset{\#}{R}(x-\ell_j)\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{jr}(\ell_j)\right]s$$
$$\times\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{jr}(\ell_j)\right]^{-1}\overset{\#}{C}{}_y^{-1}\overset{\simeq}{Y}(\ell_j-0) .$$

From Eq. (134), and since $\overset{\#}{C}{}_{12}^{-1}\overset{\#}{C}_{11}=\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}$,

$$\overset{\#}{R}(x)\overline{q}_{j+1}=\overset{\#}{R}(x-\ell_j)\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}\left[\overset{\ggg}{\overline{U}}_{jj}(\ell_j)+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\overline{U}}_{jj}(\ell_j)\right] .$$

The total wave traveling to the right on $[\ell_j, \ell_{j+1}]$, i.e., the $(j+1)^{th}$ span, can be defined as follows:

$$\overset{\gg}{\overline{U}}_{(j+1)j}(x) = \overset{\#}{R}(x)\overline{q}_{j+1}$$

$$= \overset{\#}{R}(x-\ell_j)\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}\left[\overset{\gg}{\overline{U}}_{jj}(\ell_j)+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ll}{\overline{U}}_{jj}(\ell_j)\right] \quad . \tag{138}$$

Similarly, the total wave traveling to the left on $[\ell_j, \ell_{j+1}]$, i.e., the $(j+1)^{th}$ span, is

$$\overset{\ll}{\overline{U}}_{(j+1)j}(x) = \overset{\#}{R}(\cdot x)\overset{\#}{\Gamma}_{(j+1)r}(0)\overline{q}_{j+1}$$

$$= \overset{\#}{R}(\ell_{j+1}-x)\overset{\#}{\Gamma}_{(j+1)r}(\ell_{j+1})\overset{\#}{R}(\ell_{j+1}-\ell_j)\left[I+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+1)r}(\ell_j)\right]^{-1}$$

$$\times\left[\overset{\gg}{\overline{U}}_{jj}(\ell_j)+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\ll}{\overline{U}}_{jj}(\ell_j)\right] \quad . \tag{139}$$

Hence, the dynamic response on $[\ell_j, \ell_{j+1}]$, i.e., the $(j+1)^{th}$ span, caused by driving forces $\overset{\simeq}{P}(a_{jk})$, $k = 1, 2, \ldots, k(j)$, on $[\ell_{j-1}, \ell_j]$, i.e., the j^{th} span, can be expressed by the following matrix equation:

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix}\begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix}\begin{bmatrix} \overset{\gg}{\overline{U}}_{(j+1)j}(x) \\ \overset{\ll}{\overline{U}}_{(j+1)j}(x) \end{bmatrix} \quad . \tag{140}$$

With similar mathematical manipulations, the following set of matrix equations can be obtained:

On $[\ell_{j+1}, \ell_{j+2}]$, i.e., the $(j+2)^{th}$ span,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix}\begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix}\begin{bmatrix} \overset{\gg}{\overline{U}}_{(j+2)j}(x) \\ \overset{\ll}{\overline{U}}_{(j+2)j}(x) \end{bmatrix}$$

where

$$\overset{\lll}{\overline{U}}_{(j+2)j}(x)=\overset{\#}{R}(\ell_{j+2}-x)\overset{\#}{\Gamma}_{(j+2)r}(\ell_{j+2})\overset{\#}{R}(\ell_{j+2}-\ell_{j+1})$$

$$\times\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(j+2)r}(\ell_{j+1})\right]^{-1}\left[\overset{\ggg}{\overline{U}}_{(j+1)j}(\ell_{j+1})+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\overline{U}}_{(j+1)j}(\ell_{j+1})\right] .$$

On $[\ell_{j+2}, \ell_{j+3}]$, i.e., the $(j+3)^{th}$ span,

$$\begin{matrix} \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \end{matrix}$$

$$\overset{\lll}{\overline{U}}_{(n-2)j}(x)=\overset{\#}{R}(\ell_{n-2}-x)\overset{\#}{\Gamma}_{(n-2)r}(\ell_{n-2})\overset{\#}{R}(\ell_{n-2}-\ell_{n-3})$$

$$\times\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(n-2)r}(\ell_{n-3})\right]^{-1}\left[\overset{\ggg}{\overline{U}}_{(n-3)j}(\ell_{n-3})+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\overline{U}}_{(n-3)j}(\ell_{n-3})\right] .$$

On $[\ell_{n-2}, \ell_{n-1}]$, i.e., the $(n-1)^{th}$ span,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\overline{U}}_{(n-1)j}(x) \\ \overset{\lll}{\overline{U}}_{(n-1)j}(x) \end{bmatrix}$$

where

$$\overset{\lll}{\overline{U}}_{(n-1)j}(x)=\overset{\#}{R}(\ell_{n-1}-x)\overset{\#}{\Gamma}_{(n-1)r}(\ell_{n-1})\overset{\#}{R}(\ell_{n-1}-\ell_{n-2})$$

$$\times\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_{(n-1)r}(\ell_{n-2})\right]^{-1}\left[\overset{\ggg}{\overline{U}}_{(n-2)j}(\ell_{n-2})+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\overline{U}}_{(n-2)j}(\ell_{n-2})\right] .$$

On $[\ell_{n-1}, \ell_n]$, i.e., the n^{th} span,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\overline{U}}_{nj}(x) \\ \overset{\lll}{\overline{U}}_{nj}(x) \end{bmatrix}$$

where

$$\overset{\lll}{\overline{U}}_{nj}(x)=\overset{\#}{R}(\ell_n-x)\overset{\#}{\Gamma}_n(\ell_n)\overset{\#}{R}(\ell_n-\ell_{n-1})\left[\overset{\#}{I}+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\#}{\Gamma}_n(\ell_n)\right]^{-1}$$

$$\times\left[\overset{\ggg}{\overline{U}}_{(n-1)j}(\ell_{n-1})+\overset{\#}{C}{}_{11}^{-1}\overset{\#}{C}_{12}\overset{\lll}{\overline{U}}_{(n-1)j}(\ell_{n-1})\right] \quad .$$

Again, if superposition techniques are applied, the total dynamic response on any span due to all the driving forces $\overset{\simeq}{P}(a_{jk})$, $j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, k(j)$, in traveling wave forms is as follows:

On $[\ell_{i-1}, \ell_i]$, i.e., the i^{th} span,

$$\begin{bmatrix} \overset{\simeq}{Y} \\ \overset{\simeq}{F} \end{bmatrix} = \begin{bmatrix} \frac{1}{s}\overset{\#}{C}_y & \overset{\#}{0} \\ \overset{\#}{0} & \overset{\#}{C}_f \end{bmatrix} \begin{bmatrix} \overset{\#}{C}_{11} & \overset{\#}{C}_{12} \\ \overset{\#}{C}_{21} & \overset{\#}{C}_{22} \end{bmatrix} \begin{bmatrix} \overset{\ggg}{\overline{U}}_i(x) \\ \overset{\lll}{\overline{U}}_i(x) \end{bmatrix} \tag{141}$$

where

$$\overset{\ggg}{\overline{U}}_i(x) = \sum_{j=1}^{n} \overset{\ggg}{\overline{U}}_{ij}(x) \quad ,$$

$$\overset{\lll}{\overline{U}}_i(x) = \sum_{j=1}^{n} \overset{\lll}{\overline{U}}_{ij}(x) \quad .$$

Eq. (141) is written out in detail in Chapter 2.

The subscripts, i and j, for $\overset{>}{\overline{U}}_{ij}$, $\overset{<}{\overline{U}}_{ij}$, $\overset{\gg}{\overline{U}}_{ij}$, $\overset{\ll}{\overline{U}}_{ij}$, $\overset{\ggg}{\overline{U}}_{ij}$, $\overset{\lll}{\overline{U}}_{ij}$ denote that:

If $i = j$, the wave is originated from the driving forces in the same span as where the wave is located.

If $i \neq j$, the wave is originated from the driving forces in other spans and is modified when the wave passes through supports into the span under investigation.

APPENDIX G

EVALUATION OF THE REFLECTION MATRIX OF A FIXED END

For a fixed end, at $x = 0-0$, the boundary condition is

$$\tilde{Y} = 0 \quad .$$

Hence, from Eq. (10),

$$\tilde{Y}(0) = \frac{1}{s} \overset{\#}{C}_y \left[\overset{\#}{C}_{11} \overset{\#}{R}(0) \bar{q}_1 + \overset{\#}{C}_{12} \overset{\#}{R}(-0) \bar{r}_1 \right] = 0$$

or

$$\bar{q}_1 = (-\overset{\#}{C}{}_{11}^{-1} \overset{\#}{C}_{12}) r_1 \quad . \tag{142}$$

For an end support which is not specialized, it was shown in Appendix C that

$$\bar{q}_1 = \overset{\#}{\Gamma}_0(0) \bar{r}_1 \quad .$$

Comparing this expression with Eq. (142), it can be concluded that for a fixed end at $x = 0$, the reflection matrix is

$$\overset{\#}{\Gamma}_0(0) = -\overset{\#}{C}{}_{11}^{-1} \overset{\#}{C}_{12} \quad .$$

Similarly, it can b· proved also that

$$\overset{\#}{\Gamma}_n(\ell_n) = -\overset{\#}{C}{}_{11}^{-1} \overset{\#}{C}_{12}$$

for a fixed end at $x = \ell_n$.

APPENDIX H

NOTES ON SUPPORT IMPEDANCES

To simplify the calculation of end impedance $\overset{\#}{Z}_0(0)$ or $\overset{\#}{Z}_n(\ell_n)$, where $n = 2$ for the shaft system shown in Figure 16, it can be assumed that A_0 and B_0 are rigid bodies (see Figure 1). It has been shown (reference 14) that if $\overset{\#}{Z}_c(x_j)$ is the j^{th} concentrated impedance, where $\ell_n < x_j < b$, the concentrated right end impedance $\overset{\#}{Z}_n(\ell_n)$ may be calculated by the following relationship:

$$\overset{\#}{Z}_n(\ell_n) = \sum_j \overset{\#}{S}'(\ell_n - x_j)\overset{\#}{Z}_c(x_j)\overset{\#}{S}(\ell_n - x_j), \qquad x_j \text{ on } [\ell_n, b] \tag{143}$$

Similarly, the concentrated left end impedance $\overset{\#}{Z}_0(0)$ may be calculated from

$$\overset{\#}{Z}_0(0) = \sum_k \overset{\#}{S}'(-x_k)\overset{\#}{Z}_c(x_k)\overset{\#}{S}(-x_k), \qquad x_k \text{ on } [a, 0], \tag{144}$$

where $\overset{\#}{Z}_c(x_k)$ is the k^{th} concentrated impedance, $a < x_k < 0$, and

$$\overset{\#}{S}(\gamma) = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}, \qquad \gamma \text{ is real},$$

and $\overset{\#}{S}'(\gamma)$ is the transpose of $\overset{\#}{S}(\gamma)$.

If both ends have the same configuration, i.e., $\overset{\#}{Z}_c(x_k) = \overset{\#}{Z}_c(\ell_n + |x_k|)$, where $a < x_k < 0$, Eqs. (143) and (144) can be used to calculate the concentrated end impedances as follows:

Let $\overset{\#}{Z}_c(x_k) = \overset{\#}{Z}_c(\ell_n + |x_k|) = \begin{bmatrix} Z_{c11} & 0 \\ 0 & Z_{c22} \end{bmatrix}, \qquad a < x_k < 0$.

If Eq. (143) is used and if $|x_k| = \gamma$, since x_k is a negative value, then

$$\overset{\#}{Z}_n(\ell_n) = \overset{\#}{S}'(-\gamma)\overset{\#}{Z}_c\overset{\#}{Z}(-\gamma)$$

$$= \begin{bmatrix} 1 & 0 \\ -\gamma & 1 \end{bmatrix} \begin{bmatrix} Z_{c11} & 0 \\ 0 & Z_{c22} \end{bmatrix} \begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix}$$

$$\overset{\#}{Z}_n(\ell_n) = \begin{bmatrix} Z_{c11} & -\gamma Z_{c11} \\ -\gamma Z_{c11} & \gamma^2 Z_{c11} + Z_{c22} \end{bmatrix} . \qquad (145)$$

From Eq. (144),

$$\overset{\#}{Z}_0(0) = \overset{\#}{S}'(\gamma)\overset{\#}{Z}_c\overset{\#}{S}(\gamma)$$

$$= \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} Z_{c11} & 0 \\ 0 & Z_{c22} \end{bmatrix} \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} Z_{c11} & \gamma Z_{c11} \\ \gamma Z_{c11} & \gamma^2 Z_{c11} + Z_{c22} \end{bmatrix}$$

or

$$\overset{\Delta}{Z}_0(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Z_{c11} & \gamma Z_{c11} \\ \gamma Z_{c11} & \gamma^2 Z_{c11} + Z_{c22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\overset{\Delta}{Z}_0(0) = \begin{bmatrix} Z_{c11} & -\gamma Z_{c11} \\ -\gamma Z_{c11} & \gamma^2 Z_{c11} + Z_{c22} \end{bmatrix} . \qquad (146)$$

If Eq. (145) is compared with Eq. (146), it can be concluded that

$$\overset{\Delta}{Z}_0(0) = \overset{\#}{Z}_n(\ell_n) \qquad (147)$$

provided both ends have the same configuration.

With reference to Eqs. (57),

$$\left.\begin{aligned}
\overset{\#}{\Gamma}_{1\ell}(\ell_1) &= \left[-\overset{\#}{C}_{22}+\overset{\Delta}{z}_{1\ell}(\ell_1)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\Delta}{z}_{1\ell}(\ell_1)\overset{\#}{C}_{11}\right] \\
\overset{\#}{\Gamma}_{1r}(\ell_1) &= \left[-\overset{\#}{C}_{22}+\overset{\#}{z}_{1r}(\ell_1)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\#}{z}_{1r}(\ell_1)\overset{\#}{C}_{11}\right]
\end{aligned}\right\} \tag{148}$$

$$\left.\begin{aligned}
\overset{\Delta}{z}_{1\ell}(\ell_1) &= \overset{\Delta}{z}_1(\ell_1) + \overset{\Delta}{z}_{0\ell}(\ell_1) \\
\overset{\#}{z}_{1r}(\ell_1) &= \overset{\#}{z}_1(\ell_1) + \overset{\#}{z}_{2r}(\ell_1)
\end{aligned}\right\} \tag{149}$$

$$\left.\begin{aligned}
\overset{\Delta}{z}_{0\ell}(\ell_1) &= \left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_0(\ell_1)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_0(\ell_1)\right]^{-1} \\
\overset{\#}{z}_{2r}(\ell_1) &= \left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_2(\ell_1)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]^{-1}
\end{aligned}\right\} \tag{150}$$

$$\left.\begin{aligned}
\overset{\#}{\Gamma}_0(\ell_1) &= \overset{\#}{R}(\ell_1)\overset{\#}{\Gamma}_0(0)\overset{\#}{R}(\ell_1) \\
\overset{\#}{\Gamma}_2(\ell_1) &= \overset{\#}{R}(\ell_2-\ell_1)\overset{\#}{\Gamma}_2(\ell_2)\overset{\#}{R}(\ell_2-\ell_1)
\end{aligned}\right\} \tag{151}$$

and

$$\left.\begin{aligned}
\overset{\#}{\Gamma}_0(0) &= \left[-\overset{\#}{C}_{22}+\overset{\Delta}{z}_0(0)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\Delta}{z}_0(0)\overset{\#}{C}_{11}\right] \\
\overset{\#}{\Gamma}_2(\ell_2) &= \left[-\overset{\#}{C}_{22}+\overset{\Delta}{z}_2(\ell_2)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\Delta}{z}_2(\ell_2)\overset{\#}{C}_{11}\right].
\end{aligned}\right\} \tag{152}$$

Eq. (147) implies that $\overset{\Delta}{z}_0(0) = \overset{\#}{z}_2(\ell_2)$; hence, Eq. (152) gives

$$\overset{\#}{\Gamma}_0(0) = \overset{\#}{\Gamma}_2(\ell_2) \quad . \tag{153}$$

If $\ell_2-\ell_1=\ell_1$ and if Eq. (153) is used, then, from Eq. (151),

$$\overset{\#}{\Gamma}_0(\ell_1) = \overset{\#}{\Gamma}_2(\ell_1) \quad ,$$

which also implies (see Eq. (150)) that

$$\overset{\Delta}{z}_{0\ell}(\ell_1) = \overset{\#}{z}_{2r}(\ell_1) \quad .$$

From examination of the matrix form for $\overset{\#}{Z}_1(\ell_1)$, i.e.,

$$\overset{\#}{Z}_1(\ell_1) = \begin{bmatrix} Z_{1(11)} & 0 \\ 0 & Z_{1(22)} \end{bmatrix} ,$$

it can be shown that

$$\overset{\Delta}{Z}_1(\ell_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} Z_{1(11)} & 0 \\ 0 & Z_{1(22)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} Z_{1(11)} & 0 \\ 0 & Z_{1(22)} \end{bmatrix}$$

or

$$\overset{\Delta}{Z}_1(\ell_1) = \overset{\#}{Z}_1(\ell_1)$$

which implies that $\overset{\Delta}{z}_1(\ell_1) = \overset{\#}{z}_1(\ell_1)$.

From the above relationships, it can be concluded from Eqs. (149) and (148) that

$$\overset{\#}{\Gamma}_{1\ell}(\ell_1) = \overset{\#}{\Gamma}_{1r}(\ell_1) \quad .$$

APPENDIX I

MATHEMATICAL DERIVATIONS FOR THE QUASI-MATCHED IMPEDANCE AT THE INTERIOR SUPPORT

First, all related equations pertaining to the upcoming derivations can be summarized as follows (see Eqs. (57)):

$$\overset{\#}{\Gamma}_{1r}(\ell_1)=\left[-\overset{\#}{C}_{22}+\overset{\#}{Z}_{1r}(\ell_1)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\#}{Z}_{1r}(\ell_1)\overset{\#}{C}_{11}\right] \tag{154}$$

$$\overset{\#}{Z}_{1r}(\ell_1)=\overset{\#}{Z}_{1}(\ell_1)+\overset{\#}{Z}_{2r}(\ell_1) \tag{155}$$

$$\overset{\#}{Z}_{2r}(\ell_1)=\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_2(\ell_1)\right]\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]^{-1} \tag{156}$$

$$\overset{\#}{\Gamma}_2(\ell_1)=\overset{\#}{R}(\ell_2-\ell_1)\overset{\#}{\Gamma}_2(\ell_2)\overset{\#}{R}(\ell_2-\ell_1) \tag{157}$$

$$\overset{\#}{\Gamma}_2(\ell_2)=\left[-\overset{\#}{C}_{22}+\overset{\#}{Z}_{2}(\ell_2)\overset{\#}{C}_{12}\right]^{-1}\left[\overset{\#}{C}_{21}-\overset{\#}{Z}_{2}(\ell_2)\overset{\#}{C}_{11}\right] \tag{158}$$

where

$$\overset{\#}{Z}_2(\ell_2)=\begin{bmatrix} z_{2(11)} & z_{2(12)} \\ z_{2(21)} & z_{2(22)} \end{bmatrix}$$

$$\overset{\#}{\Gamma}_2(\ell_2)=\begin{bmatrix} \Gamma_2(\ell_2)_{11} & \Gamma_2(\ell_2)_{12} \\ \Gamma_2(\ell_2)_{21} & \Gamma_2(\ell_2)_{22} \end{bmatrix} .$$

If direct matrix algebraic operations are used in Eq. (158),

$$\overset{\#}{\Gamma}_2(\ell_2)=\frac{1}{\det\left[-\overset{\#}{C}_{22}+\overset{\#}{Z}_{2}(\ell_2)\overset{\#}{C}_{12}\right]} \times \begin{bmatrix} e_2e_3-ie_2z_{2(21)}+ie_3z_{2(22)} & -1+ie_2z_{2(11)}-ie_3z_{2(12)} \\ e_1+ie_1e_3z_{2(21)}+z_{2(22)} & -ie_3-ie_1e_3z_{2(11)}-z_{2(12)} \end{bmatrix}$$

$$\times \begin{bmatrix} -ie_3+ie_1e_3z_{2(11)}-z_{2(12)} & 1+ie_2z_{2(11)}+ie_3z_{2(12)} \\ e_1+ie_1e_3z_{2(21)}-z_{2(22)} & -e_2e_3+ie_2z_{2(21)}+ie_3z_{2(22)} \end{bmatrix} . \tag{159}$$

Examination of Eqs. (156) and (157) shows that

$$\overset{\#}{\Gamma}_2(\ell_1) = \overset{\#}{R}(\ell_2-\ell_1)\overset{\#}{\Gamma}_2(\ell_2)\overset{\#}{R}(\ell_2-\ell_1)$$

$$\begin{bmatrix} e^{-ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}e^{-ie_1\sqrt{\omega}\,(\ell_2-\ell_1)} & e^{-e_2\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{12}e^{-e_1\sqrt{\omega}\,(\ell_2-\ell_1)} \\ e^{-ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{21}e^{-e_2\sqrt{\omega}\,(\ell_2-\ell_1)} & e^{-e_2\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{22}e^{-e_2\sqrt{\omega}\,(\ell_2-\ell_1)} \end{bmatrix} .$$

The only element not being modified by an exponential decaying function in the above matrix is the one in the first row and the first column. For quasi-matching, only the element not being modified by an exponential decaying term needs to be considered; i.e.,

$$\overset{\#}{\Gamma}_2(\ell_1) = \overset{\#}{R}(\ell_2-\ell_1)\overset{\#}{\Gamma}_2(\ell_2)\overset{\#}{R}(\ell_2-\ell_1)$$

$$\approx \begin{bmatrix} e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11} & 0 \\ 0 & 0 \end{bmatrix} .$$

From Eq. (159),

$$\Gamma_2(\ell_2)_{11} = \frac{1}{\det\left[-\overset{\#}{C}_{22}+\overset{\#}{Z}_2(\ell_2)\overset{\#}{C}_{12}\right]}\Big[(1+e_3^2)(z_{2(22)}+ie_1e_2z_{2(11)}) + (e_1e_3^2+ie_2)(z_{2(12)}z_{2(21)}-z_{2(11)}z_{2(22)}) - e_3(e_2+ie_1)(z_{2(12)}+z_{2(21)})-(e_1+ie_2e_3^2)\Big] , \tag{160}$$

where

$$\det\left[-\overset{\#}{C}_{22}+\overset{\#}{Z}_2(\ell_2)\overset{\#}{C}_{12}\right]$$

$$= -(ie_3+ie_1e_3z_{2(11)}+z_{2(12)})(e_2e_3-ie_2z_{2(21)}+ie_3z_{2(22)}) +$$

$$(e_1+ie_1e_3z_{2(21)}+z_{2(22)})(1-ie_2z_{2(11)}+ie_3z_{2(12)}) \quad .$$

With reference to Eq. (156),

$$\left[\overset{\#}{C}_{21}+\overset{\#}{C}_{22}\overset{\#}{\Gamma}_2(\ell_1)\right]$$

$$= \begin{bmatrix} -ie_3 & 1 \\ e_1 & -e_2e_3 \end{bmatrix} + \begin{bmatrix} ie_3 & -1 \\ e_1 & -e_2e_3 \end{bmatrix}\begin{bmatrix} e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} ie_3\left(e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11} - 1\right) & 1 \\ e_1\left(e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11} + 1\right) & -e_2e_3 \end{bmatrix} .$$

Similarly,

$$\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]$$

$$= \begin{bmatrix} -ie_1e_3 & -ie_2 \\ 1 & -ie_3 \end{bmatrix} + \begin{bmatrix} -ie_1e_3 & -ie_2 \\ -1 & ie_3 \end{bmatrix}\begin{bmatrix} e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -ie_1e_3\left(e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11} + 1\right) & -ie_2 \\ -\left(e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11} - 1\right) & -ie_3 \end{bmatrix} ,$$

or

$$\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right] = -e_1e_3^2\left(e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}+1\right) - ie_2\left(e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}-1\right) ,$$

then,

$$\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]^{-1} = \frac{1}{\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]}$$

$$\times\begin{bmatrix} -ie_3 & ie_2 \\ \left(e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}-1\right) & \left(-ie_1e_3\,e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}+1\right)\end{bmatrix}.$$

Hence,

$$\overset{\#}{z}_{2r}(\ell_1) = \frac{1}{\det\left[\overset{\#}{C}_{11}+\overset{\#}{C}_{12}\overset{\#}{\Gamma}_2(\ell_1)\right]}$$

$$\times\begin{bmatrix} (1+e_3^2)\left(e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}-1\right) & e_3(e_2+ie_1)e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}+e_3(e_2-ie_1) \\ -e_3(e_2+ie_1)e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}+e_3(e_2-ie_1) - & ie_1e_2(1+e_3^2)\left(e^{-2ie_1\sqrt{\omega}\,(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}+1\right)\end{bmatrix}.$$

From Eq. (155),

$$\overset{\#}{z}_{1r}(\ell_1) = \overset{\#}{z}_1(\ell_1) + \overset{\#}{z}_{2r}(\ell_1) ,$$

or

$$\begin{bmatrix} z_{1r(11)} & z_{1r(12)} \\ z_{1r(21)} & z_{1r(22)}\end{bmatrix} =$$

$$
\left[\begin{array}{l}
z_{1(11)}+\dfrac{(1+e_3^2)}{\det\left[C^{\#}_{11}+C^{\#}_{12}\Gamma^{\#}_2(\ell_1)\right]}\left(e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}-1\right) \\[2ex]
\dfrac{e_3}{\det\left[C^{\#}_{11}+C^{\#}_{12}\Gamma^{\#}_2(\ell_1)\right]}\left[-(e_2+ie_1)e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}+(e_2-ie_1)\right]
\end{array}\right.
$$

$$
\left.\begin{array}{l}
\dfrac{e_3}{\det\left[C^{\#}_{11}+C^{\#}_{12}\Gamma^{\#}_2(\ell_1)\right]}\left[-(e_2+ie_1)e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}+(e_2-ie_1)\right] \\[2ex]
z_{1(22)}+\dfrac{ie_1e_2(1+e_3^2)}{\det\left[C^{\#}_{11}+C^{\#}_{12}\Gamma^{\#}_2(\ell_1)\right]}\left(e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}+i\right)
\end{array}\right] .
$$

This relationship gives the expression for each element of $\overset{\#}{z}_{1r}(\ell_1)$, since two matrices are equal to each other if and only if each corresponding element of these two matrices is identical. Thus,

$$
\left.\begin{array}{l}
z_{1r(11)}=z_{1(11)}+\dfrac{(1+e_3^2)}{\det\left[C^{\#}_{11}+C^{\#}_{12}\Gamma^{\#}_2(\ell_1)\right]}\left(e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}-1\right) \\[2ex]
z_{1r(22)}=z_{1(22)}+\dfrac{ie_1e_2(1+e_3^2)}{\det\left[C^{\#}_{11}+C^{\#}_{12}\Gamma^{\#}_2(\ell_1)\right]}\left(e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}+1\right) \\[2ex]
z_{1r(12)}=z_{1r(21)}=\dfrac{e_3}{\det\left[C^{\#}_{11}+C^{\#}_{12}\Gamma^{\#}_2(\ell_1)\right]} \\[2ex]
\qquad\times\left[-(e_2+ie_1)e^{-2ie_1\sqrt{\omega}(\ell_2-\ell_1)}\Gamma_2(\ell_2)_{11}+(e_2-ie_1)\right] .
\end{array}\right\} \quad (161)
$$

Similar to the way Eqs. (158) and (159) are solved, Eq. (154) can be expanded in complete matrix form, as follows:

$$\overset{\#}{\Gamma}_{1r}(\ell_1) = \frac{1}{\det\left[-\overset{\#}{C}_{22}+\overset{\#}{z}_{1r}(\ell_1)\overset{\#}{C}_{12}\right]}$$

$$\times \begin{bmatrix} e_2e_3-ie_2z_{1r(21)}+ie_3z_{1r(22)} & -1+ie_2z_{1r(11)}-ie_3z_{1r(12)} \\ e_1+ie_1e_3z_{1r(21)}+z_{1r(22)} & -ie_3-ie_1e_3z_{1r(11)}-z_{1r(12)} \end{bmatrix}$$

$$\times \begin{bmatrix} -ie_3+ie_1e_3z_{1r(11)}-z_{1r(12)} & 1+ie_2z_{1r(11)}+ie_3z_{1r(12)} \\ e_1+ie_1e_3z_{1r(21)}-z_{1r(22)} & -e_2e_3+ie_2z_{1r(21)}+ie_3z_{1r(22)} \end{bmatrix} .$$

Hence, the quasi-matching condition $\Gamma_{1r}(\ell_1)_{11} = 0$ gives Eq. (62) in the text where $z_{1r(11)}$, $z_{1r(12)}$, $z_{1r(21)}$, $z_{1r(22)}$ can be calculated from Eq. (161).

APPENDIX J

MATHEMATICAL DERIVATIONS OF THE IMPEDANCE OF A FLOATING RING DAMPER ASSEMBLY AS THE INTERIOR SUPPORT

With reference to Figure 18, a free-body diagram can be drawn for an infinitesimal mass element at $x = \ell_1$, as shown in Figure 25. When the equilibrium condition at point (1) is considered,

$$K_c Y_c = C(Y_{bt} - Y_{ct}) + K_b(Y_b - Y_c) \quad ,$$

of which the Laplace transform is

$$K_c \tilde{Y}_c = sC(\tilde{Y}_b - \tilde{Y}_c) + K_b(\tilde{Y}_b - \tilde{Y}_c)$$

or

$$\tilde{Y}_c = \frac{sC+K_b}{sC+K_b+K_c} \tilde{Y}_b \quad . \tag{162}$$

When the equilibrium condition at point (2) is considered,

$$K_a(Y_1 - Y_b) = M_b Y_{btt} + C(Y_{bt} - Y_{ct}) + K_b(Y_b - Y_c) \quad ,$$

of which the Laplace transform is

$$K_a(\tilde{Y}_1 - \tilde{Y}_b) = s^2 M_b \tilde{Y}_b + sC(\tilde{Y}_b - \tilde{Y}_c) + K_b(\tilde{Y}_b - \tilde{Y}_c) \quad .$$

If Eq. (162) is substituted into the above expression,

$$\tilde{Y}_b = \frac{K_a[sC+(K_b+K_c)]}{s^3 M_b C + s^2 M_b(K_b+K_c) + sC(K_a+K_c) + (K_a K_b + K_b K_c + K_c K_a)} \tilde{Y}_1 \quad . \tag{163}$$

When the equilibrium condition at point (3) is considered,

$$-\overline{R}(\ell_1) = \begin{bmatrix} M_a Y_{1tt} + K_a(Y_1 - Y_b) \\ \\ 0 \end{bmatrix} ,$$

of which the Laplace transform is

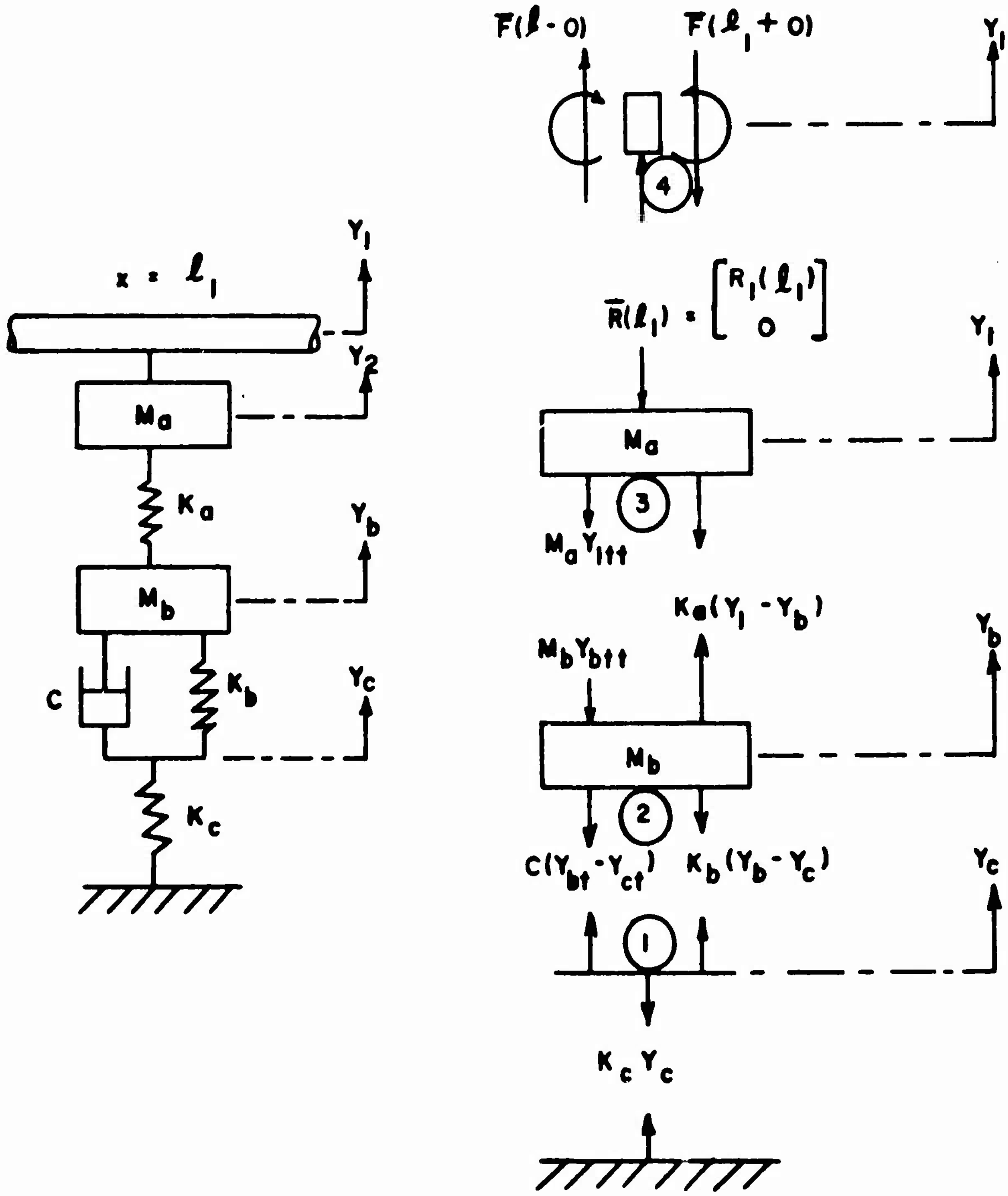

Figure 25. Free-Body Diagram for Forces Existing at $x = \ell_1$ of a Floating Ring Damper Assembly.

$$-\overset{\simeq}{R}(\ell_1) = \begin{bmatrix} (s^2M_a+K_a)\tilde{Y}_1 - K_a\tilde{Y}_b \\ 0 \end{bmatrix} .$$

If Eq. (163) is substituted into the above expression,

$$-\overset{\simeq}{R}(\ell_1) = \begin{bmatrix} \dfrac{s^5M_aM_bC+s^4M_aM_b(K_b+K_c)+s^3[M_a(K_a+K_c)+M_bK_a]C+s^2[(M_a+M_b)K_a(K_b+K_c)+M_aK_bK_c]+sK_aK_cC+K_aK_bK_c}{s^3M_bC+s^2M_b(K_b+K_c)+s(K_a+K_c)C+(K_aK_b+K_bK_c+K_cK_a)}\tilde{Y}_1 \\ 0 \end{bmatrix} . \tag{164}$$

When the equilibrium condition at point 4 is considered,

$$\overset{\simeq}{F}(\ell_1-0) - \overset{\simeq}{F}(\ell_1+0) = -\overset{\simeq}{R}(\ell_1) .$$

If Eq. (164) is substituted into the above equation,

$$\overset{\simeq}{F}(\ell_1-0) - \overset{\simeq}{F}(\ell_1+0)$$

$$= s\begin{bmatrix} \dfrac{s^4M_aM_bC+s^3M_aM_b(K_b+K_c)+s^2[M_a(K_a+K_c)+M_bK_a]C+s[(M_a+M_b)K_a(K_b+K_c)+M_aK_bK_c]+K_aK_cC+\frac{1}{s}K_aK_bK_c}{s^3M_bC+s^2M_b(K_b+K_c)+s(K_a+K_c)C+(K_aK_b+K_bK_c+K_cK_a)} & 0 \\ 0 & 0 \end{bmatrix}\overset{\simeq}{Y}(\ell_1) .$$

If the above expression is compared with

$$\overset{\simeq}{F}(\ell_1-0) - \overset{\simeq}{F}(\ell_1+0) = s\overset{\#}{Z}_1(\ell_1)\overset{\simeq}{Y}(\ell_1) \quad ,$$

it can be concluded that the support impedance at $x = \ell_1$ is

$$\overset{\#}{Z}_1(\ell_1)$$

$$= \begin{bmatrix} \dfrac{s^4M_aM_bC+s^3M_aM_b(K_b+K_c)+s^2[M_a(K_a+K_c)+M_bK_a]C + s[(M_a+M_b)K_a(K_b+K_c)+M_aK_bK_c]+K_aK_cC+\frac{1}{s}K_aK_bK_c}{s^3M_bC+s^2M_b(K_b+K_c)+s(K_a+K_c)C+(K_aK_b+K_bK_c+K_cK_a)} & 0 \\ 0 & 0 \end{bmatrix} . \quad (165)$$

DISTRIBUTION

US Army Materiel Command	5
US Army Aviation Materiel Command	5
US Army Aviation Materiel Laboratories	13
US Army R&D Group (Europe)	2
US Army Research Office-Durham	1
US Army Test and Evaluation Command	1
US Army Medical R&D Command	1
US Army Combat Developments Command, Fort Belvoir	2
US Army Aviation School	1
US Army Infantry Center	2
US Army Tank-Automotive Center	2
US Army Armor and Engineer Board	1
US Army Aviation Test Activity, Edwards AFB	2
Air Force Flight Test Center, Edwards AFB	1
US Army Field Office, AFSC, Andrews AFB	1
Systems Engineering Group (RTD), Wright-Patterson AFB	2
Systems Engineering Group (SEFDP), Wright-Patterson AFB	1
Air Force Flight Dynamics Laboratory, Wright-Patterson AFB	1
Air Force Aero Propulsion Laboratory, Wright-Patterson AFB	1
Naval Air Systems Command	4
Chief of Naval Research	3
Marine Corps Liaison Officer, US Army Transportation School	1
Ames Research Center, NASA	1
Lewis Research Center, NASA	1
NASA Scientific and Technical Information Facility	2
NAFEC Library (FAA)	2
Defense Documentation Center	20
US Government Printing Office	1

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) University of Virginia School of Engineering and Applied Science Charlottesville, Virginia 22903	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP
3. REPORT TITLE Analysis of the Dynamics of Supercritical Shafts on Many Flexible Supports Using Transmission Line Analogy	
4. DESCRIPTIVE NOTES (Type of report and inclusive dates)	
5. AUTHOR(S) (Last name, first name, initial) Liu, Y. N.; and Friedericy, J. A.	

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1966	176	21

8a. CONTRACT OR GRANT NO. AMC(T)-01-66-16 b. PROJECT NO. c. Task No. 1M121401D14414 d	9a. ORIGINATOR'S REPORT NUMBER(S) USAAVLABS Technical Report 66-44 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
10. AVAILABILITY/LIMITATION NOTICES Distribution of this document is unlimited.	
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Department of the Army U.S. Army Aviation Materiel Laboratories Fort Eustis, Virginia 23604

13. ABSTRACT

This report presents an analysis of the dynamics of supercritical shafts on many flexible supports in terms of a so-called "transmission line analogy." The primary objective of the study is to develop a direct analytical approach for optimizing the support conditions, in terms of minimum flexural vibration behavior, for supercritical shafts flexibly supported on mass-spring-damper units at different locations along the shaft. The approach is based on the traveling wave concept as used in electrical transmission line theory. The governing differential equation used in this analogy includes terms which account for the effects of rotating inertia, gyroscopic motion, and shear deformations.

If the solution of the governing differential equation is manipulated by means of rather complicated matrix algebra, the dynamic response of the rotating shaft can be expressed in traveling wave form, which leads to the criterion for impedance matching and optimized support conditions. The impedance at ach supporting location corresponding to minimum vibration response must equal the characteristic impedance of the shaft. This condition is termed a "matched" condition.

A weaker than optimum form of impedance matching is the "quasi-matched" condition, in which only the predominant term of the reflection matrix for a support is made to vanish.

A rotating shaft with three supports is used to illustrate the matched impedance concept for determining optimum support conditions for multisupported hypercritical shafts.

DD FORM 1 JAN 64 1473

UNCLASSIFIED
Security Classification

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
supercritical shafts						
electrical transmission line analogy						
minimum flexural vibration behavior						
traveling wave concept						
impedance matching						
optimized support conditions						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.

UNCLASSIFIED
Security Classification

6048-66